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SCHEDULING MAINTENANCE OPERATIONS WHICH

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BY

BEHNAM EBRAHIMIAN AND LEONARD SHAW

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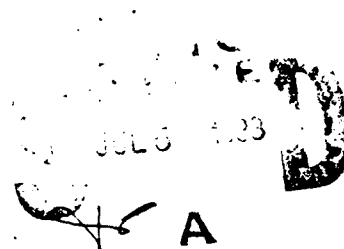CAUSE AGE-DEPENDENT FAILURE RATE CHANGES

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REPORT DOCUMENTATION PAGE	READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER	2. GOVT ACCESSION NO. AD-A130076	3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)	5. TYPE OF REPORT & PERIOD COVERED
Scheduling Maintenance Operations Which Cause Age-Dependent Failure Rate Changes	
	6. PERFORMING ORG. REPORT NUMBER Poly-EE/CS-83-002

7. AUTHOR(s)	8. CONTRACT OR GRANT NUMBER(s)
Behnam Ebrahimian & Leonard Shaw	N00014-75-C-0858

9. PERFORMING ORGANIZATION NAME AND ADDRESS	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
Dept. of Electrical Engineering & Computer Science Polytechnic Institute of New York 333 Jay St., Brooklyn, N.Y. 11201	

11. CONTROLLING OFFICE NAME AND ADDRESS	12. REPORT DATE June 1, 1983
Office of Naval Research Arlington, Virginia 22217	13. NUMBER OF PAGES 116

14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)	15. SECURITY CLASS. (of this report) Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Reproduction in whole or in part is permitted for any purpose of the United States Government

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Maintenance schedules, failure rate, age replacement, optimal scheduling

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report studies the optimization of schedules for maintenance or repairs, for repairable stochastically failing systems, with conditional lifetime distribution function $F_i(t|\theta_i)$ and failure rate $r_i(t|\theta_i)$, where θ_i describes the system's aging history at the time of the ith repair. The novelty here is that the failure rate after a maintenance operation is a function of the sytem's previously expended lifetime. This generalizes earlier work by others on the simpler case where the future failure rate depends on the number of previous

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20. ABSTRACT (cont'd.)

repairs, but not on the times when they took place. Two major preventive strategies are considered: (1) age replacement or policy I; (2) periodic replacement with minimal repair at failure or policy II. ⟵⎯⎯⎯

The objective for each policy is to find a set of successive maintenance intervals $T_1, T_2, \ldots, T_N$, and the number N, such that the expected long-term average cost per unit time is minimized. Methodologically, the thesis employs dynamic, nonlinear programming and simulation techniques to optimize these two strategies, but the focus is on the nonlinear programming and developing efficient algorithms.

For policy I, two models are developed. The first case is appropriate when a simulation technique is to be employed to opitmize this strategy. In the second case, an approximation to the cost function is suggested and relations satisfying the necessary condition of the optimality are derived. For vaious models of policy I, it is shown that there exist lower bounds to the optimum numbers of preventive maintenance operations N^*, which can be utilized as starting search values.

Optimal policy II is analyzed for two types of failure rates:
(1) $r_i(t|\theta_{i-1}) = \theta_{i-1} r(t)$,   (2) $r_i(t|\theta_{i-1}) = r(t) + \theta_{i-1}$. In each case, relations satisfying the necessary and sufficient conditions of the optimality are derived and properties of the optimal solutions are proved.

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CHAPTER 1

AN INTRODUCTION TO PREVENTIVE MAINTENANCE POLICIES

1.1 INTRODUCTION

During the past two decades, there has been a continuing interest in studying maintenance models for systems with stochastic failures and in their military, industrial, environmental, and ecological applications. These applications may include the maintenance of complex electronic and/or mechanical equipment such as computers and airplanes, control of pollutants in the environment, maintenance of the human body, and maintenance of ecological balance in populations of plants and animals.

Barlow and Proschan (1965), McCall (1965), and Pierskalla and Voelker (1973) have researched and surveyed various maintenance models in the area of maintainability.

A broad class of what in the area of maintainability is known as optimal preventive maintenance models has received the most attention in the literature. An equipment whose failure rate increases with time (e.g. Weibull, truncated normal, gamma distributions) may break down during its actual operation more frequently as it ages. These break-downs can often be costly and a preventive maintenance strategy can cut down on the irregularity of breakdowns by carrying out regularly planned repairs before the system fails. A preventive maintenance policy can also determine the replacement time after which the existing system is no longer economical to keep. To judge the quality of main-

tenance schedules, the decision can be based on several economic criteria, including minimizing the long-term average cost per unit time and maximizing the steady state availability or limiting efficiency which is the fraction of up-time over a long time interval.

A substantial body of literature on the preventive maintenance of stochastically failing systems has been devoted to the analysis of two strategies. One is called "Age Replacement" or policy I, and the other is known as "Periodic Replacement with Minimal Repair at Failure" or policy II. It is usually assumed that the failure rate is a strictly increasing function of time. In earlier works, generally, a preventive maintenance action refers to a renewal where the system becomes as good as new. Another assumption that seems to be made in the literature most frequently is the notion of zero down-time duration for either repair or replacement.

Here, we present a brief discussion and literature review on Policies I and II.

1.2 POLICY I

This strategy is most useful in maintaining less complex equipment, such as car engines. This strategy is defined as follows:

Perform preventive maintenance at the time of failure or if T hours of actual operation have elapsed. Preventive maintenance is then rescheduled. This policy is also referred to as "Age Replacement Policy."

Barlow and Hunter (1960) have studied this preventive maintenance policy. They discussed the existence and uniqueness of the optimum cycle T^*, which is the time between two successive scheduled mainte-

nance actions. Their objective is to maximize the limiting efficiency (steady state availability) if the unit was replace at failure or at T^*, whichever occurred first. They have also shown that the optimum maintenance time is the solution to an integral equation.

Barlow and Proschan (1965) have an equivalent model but their criterion is to minimize the long-term expected average cost per unit time, which is the expected total cost for a cycle divided by the length of that cycle.

Glasser (1967) has obtained solutions to the age replacement problem for Weibull, gamma, and truncated normal distributions.

Fox (1966) has used a total discounted cost criterion to optimize the type I policy. He has shown that the optimum age of the unit T^* is a solution to an integral equation.

Most age replacement policies presented in the literature consider two costs for replacing failed and non-failed units. Scheaffer (1971) extends the standard age replacement model by including an age-dependent maintenance cost. This cost may reflect the increase in total cost due to depreciation, wear, age, diminishing productivity or reduction in salvage value of the equipment. He proposed two cost functions for which the failure rate may also be constant in order to find an optimum policy I.

In practice, the maintenance is not necessarily the replacement of the whole system, but is often the repair or replacement of part of the system. This kind of maintenance action does not always renew the system and the mean lifetime of the repaired system is usually less than the new one. Tahara and Nishida (1973) proposed models in which the system cannot recover completely after each repair. This notion has

been used by Nguyen and Murthy (1981). They have studied policy I and policy II with a basic assumption that, after each repair, the life-time distribution of the system changes in such a way that its failure rate becomes an increasing function of the number of previous repairs.

1.3 POLICY II

This strategy is applied to very expensive and complex equipment such as computers and airplanes, and it is defined as follows:

Perform preventive maintenance on the system after it has been operating a total of T hours regardless of the number of intervening failures. Preventive maintenance is then re-scheduled. When a failure occurs, it is corrected by a minimal repair.

This strategy is also known as "Periodic Replacement with Minimal Repair at Failure." The action of restoring a failed system to operation without affecting its failure rate is called minimal repair. The system is usually completely renewed after a fixed number of the periodic main-tenance cycles.

Most preventive maintenance policies presented in the literature have studied the type II policy together with policy I, including Barlow and Hunter (1961) and Barlow and Proschan (1965).

Bellman (1865) and Descamp (1965) applied dynamic programming to policy II described above.

It is assumed, generally, that minimal repairs performed, when policy II is followed, have zero time duration. But Sivazlian (1973) has generalized this model by permitting a random down-time duration for each minimal repair. This random time has an arbitrary distribution.

His criterion is to minimize the long-term expected total cost. He also derives the necessary and sufficient conditions for which this policy II is optimal.

Makabe and Morimura (1963a, 1963b, and 1965) and Morimura (1970) have studied another similar policy (policy III) along with policies I and II. Under policy III, the system is replaced at the Kth failure. The intervening failures are corrected by minimal repairs. They have shown the existence of the optimal policy III both with respect to limiting efficiency and another criterion which they called the "M i-tenance Cost Rate." It is defined as follows:

[Cost per unit down time] × [expected fraction of down-time]
+ [expected cost of all repairs and replacements during a unit time]

Morimura (1970) has studied a more general policy (IV). Under this policy, preventive maintenance is performed when at the time of failure either the total operating time has exceeded t^* or when K^{th} failure occurs. Other failures are corrected with minimal repairs. Note that when K approaches infinity, the preventive maintenance is performed at t^* and consequently we obtain the type II policy. When $t^* = 0$, then the preventive maintenance is carried out at the Kth failure and we lead to the type III policy.

As mentioned earlier, the works presented by Tahara and Nishida (1973) and Nguyen and Murthy (1981) are of great importance in studying policies I and II. The former introduces models in which the system does not recover completely after each repair, and the latter relates the repaired system's failure rate to the number of repairs.

1.4 AN OVERVIEW

We consider optimization of schedules for maintenance or repairs when two classes of maintenance strategies, namely I and II, are of interest. These policies are said to be optimum if they minimize the expected long-term average cost per unit time.

Under policy I, repairs are performed at failure or at T_i units of time, measured from the $(i-1)^{st}$ maintenance operation, and replacement occurs at t_N defined by

$$t_N = \sum_{i=1}^{N} \min (T_i, \tau_i) \qquad (1.1)$$

where τ_i is a random variable representing the failure time from the most recent maintenance action. There is only one kind of repair associated with this policy, by which the system improves but does not recover completely. Our basic assumption is that the lack of complete recovery is due to the system's previously expended lifetime.

Under policy II, there are two kinds of repairs:

1. Minor repairs to restore a failed system to operation without affecting its failure rate.

2. Major repairs to reduce the future number of failures.

Obviously, in this model the only repair that may be planned is the major one by which the failure rate decreases but the system will not become as good as new. The reason that the regeneration does not occur could be based on the previous number of repairs, age, inefficiency of repairs, and many other factors. However, our main assumption is that the system's age plays a significant role on the recovery after a major repair. Here, major repairs are scheduled at $T_1, T_2, \ldots, T_N$ and replacement at

$$t_N = \sum_{i=1}^{N} T_N \qquad (1.2)$$

The novelty here is that the failure rate after a maintenance operation is a function of the system's previously expended lifetime. This generalizes earlier work by Nguyen and Murthy (1981) on the simpler case where the future failure rate depends on number of previous repairs, but not on the times when they took place.

Our goal is to optimize both policies I and II, separately, by finding a set of successive time intervals $T_1, T_2, \ldots, T_N$ and the number N, where a replacement is made at t_N given by relations (1.1) and (1.2).

These models are only applicable when the system's failure rate is increasing with time. All of our results are valid when the underlying distribution is Weibull, whereas most can be applied to a broad class of increasing hazard rate equipments. The failure characteristic of the system at the beginning of the ith time interval is described by a conditional distribution function $F_i(t|\theta_{i-1})$, where θ_{i-1} describes the previous history of the system. Our contribution can be characterized by letting θ_{i-1} represent the age factor affecting the system's failure rate. Policies I and II are studied in Chapters 2 and 3 respectively.

Section 2.2 introduces policy I, which is a generalization of "repair at failure or at at time T, whichever comes first" to reduce the irregularity of breakdowns, assumptions involving zero time duration for a repair or replacement, and constant replacement. Repair and breakdown costs are also discussed in this section.

Since we assume that the recovery after each repair is not fully achieved, function θ_i is defined in Section 2.3 to relate the system's age and the number of repairs to the failure rate. A formulation of the

long-term expected average cost per unit time for policy I is also found in this section.

Section 2.4 contains a general formulation of the type I policy as a nonlinear programming problem (NPI) and a discussion on three special cases:

1.   Replacement at time T or failure, whichever occurs first.

2.   The system ages rapidly or repairs are destructive ($\theta_1 >> \theta_0$).

3.   The system ages slowly ($\frac{\theta_i - \theta_{i-1}}{\theta_{i-1}}$ is small)

A reformulation which is suitable to dynamic programming and its computational efficiency is discussed in Section 2.5.

Section 2.6 is concerned with the state dependent model where the state of the system is defined to be the number of previous repairs performed on that system. In other words, after each repair, the new failure rate will depend on the number of previous repairs caused by either failure or planned preventive maintenance operation. A review of this model, which was originally developed by D.G. Nguyen and Murthy (1981), along with assumptions, uniqueness and existence of the optimal solution and a computational algorithm are described in this section. This computational algorithm becomes inefficient when the number of time intervals in each renewal cycle is large. Lemma 2.1 and policy I' will permit us to develop a heuristic procedure which is computationally more efficient. This is achieved by finding a good estimate for N*, the optimum number of planned repairs in each renewal cycle. A numerical example for systems with Weibull lifetime distribution and an application to lemma 2.1 are also presented.

Our work extends the maintenance models to an age-dependent case. In other words, the amount by which recovery after a repair is incom-

plete is related to the length of previous operating time. Section 2.7 contains two suggestions on how that age factor can be incorporated into the analysis of policy I.

1. Since the system's age at the time of repair is a random variable and, consequently, it will complicate the derivation of the long-term cost per unit time, a simulation approach seems to be more efficient to optimize the age-dependent type I policy. When age factor after the ith repair is considered to have the form

$$\theta_i = 1 + \epsilon \{\text{age at the time of the } i^{th} \text{ repair}\}$$

where   is a constant age deterioration factor, and it measures how fast the system deterioriates with respect to its age.

2. To have a model that will be analytically tractable, we may approximate the age by its expected value. To be specific, the age factor in this case at the time of the ith preventive maintenance action can be defined as

$$\theta_i = 1 + \epsilon \{\text{expected age at the time of the } i^{th} \text{ repair}\}$$

where the age deterioration factor, ϵ, is a constant parameter which determines the relation between the expected age and the deterioration level of the system.

Section 2.8 deals only with the simulation approach. A general type I policy is reformulated such that the resulting expected long-term average cost per unit time is consistent with the previous formulation derived in Section 2.3. This mathematical relation is used to generate the cost function by means of random numbers. Among various models of this strategy, simulation is made of a case in which the age at the

time of repair will determine the new failure rate. Also, its optimization with respect to the expected long-term average cost criterion is discussed. Efficiency and reduction in computation time are obtained by defining a relation among $T_1, T_2, \ldots, T_N$ and by estimating the optimum number of preventive maintenance operations in each renewal cycle.

In Section 2.9, we develop an approximate analytical model for the age-dependent type I policy. The approximation is done by replacing the system's age, at the time of the ith repair, by its expected value. A system of nonlinear equations which satisfy the necessary conditions of optimality will be derived. A procedure to compute gradients is developed so that the optimum solutions can be found by applying a numerical search technique. It is also shown that, when underlying lifetime distribution is Weibull, computation of gradients can be more simplified.

In Section 2.10, policy I is formulated in terms of steady state availability.

Chapter 3 is entirely devoted to policy II and its optimalities.

Section 3.2 contains an introduction to policy II by which planned major repairs are performed at T_i; $i = 1, 2, \ldots, N$ with minor repairs at failures without disturbing the failure rates. The replacement takes place at $t_N = \sum\limits_{i=1}^{N} T_i$.

Different assumptions regarding the hazard rate allows us to define two general models in Section 3.3. "Model A" is applied when the failure rate, after each repair, returns to the same value as at the time of replacement but with a larger slope than if replacement had taken place. "Model B" deals with systems for which, after each repair, the failure rate declines with unchanged slope, but it does not necessarily return to the point it started from.

In Section 3.4, a general mathematical representation of the expected long-term average cost per unit time is derived for policy II.

Optimal policy II is the solution to a nonlinear programming problem (NPII) which is found in Section 3.5. First three special cases are discussed.

1. Replacement at time T (constant) with minor repairs at failures. A solution to this problem for Weibull lifetime distributions is also obtained.

2. The system ages rapidly.

3. The system ages slowly.

In section 3.6, a dynamic programming formulation to type II policy is defined and its numerical computational efficiency is described.

To ensure the convergence of various optimum search techniques applied to this problem, we investigate the existence of the minimum point in Section 3.7. Along with defining some notations, relations satisfying the necessary condition of the optimality are also derived. Theorem 3.1 discusses the conditions under which the minimum point can exist. Proper initial values for searching are obtained by using Theorem 3.2.

Section 3.8 contains a review of the state-dependent model developed by Nguyen and Murthy (1981). This model assumes that the number of repairs affects the future failure rates.

Section 3.9 defines two forms of failure rates by which the system's age can be incorporated into the type II policy. One is called the A-type model with the age factor represented as

$$\theta_i = 1 + \sum_{j=1}^{i} T_j \qquad (1.3)$$

The other is the B-type model with the age factor represented as

$$\theta_i = \sum_{j=1}^{i} T_j$$

Section 3.10 is devoted to the optimality of A-type age-dependent policy II. Necessary conditions for optimality are derived for both general and Weibull distributions. Theorem 3.3 describes the order of the optimal scheduled repair times when the underlying lifetime distributions have the Weibull form. Other properties of the optimal solution are exemplified.

The age-dependent policy II of type B and its optimality is found in Section 3.11. The analysis involves the formulation, necessary and sufficient conditions of the optimality. It is also shown that for a fixed value of N, the optimal repair time is a solution to an integral equation. Theorem 3.4 proves a useful inequality relating the optimum scheduled repair times and the number of major repairs in each renewal cycle. This theorem is then used to find a simpler form for the sufficient conditions when the underlying lifetime distribution has the Weibull form. The optimum number of the planned repair times N^*, is computed in terms of optimum scheduled repair times, and it is used to obtain a lower bound for N^* in terms of maintenance costs and the failure distribution parameters (e.g. Weibull). Two analytical solutions are derived for two cases of Weibull failure distribution.

In Section 3.12 the policy II is formulated in terms of steady state availability.

The major results and general conclusions are summarized in Chapter 4.

CHAPTER 2

POLICY I
SCHEDULING MAXIMUM TIMES
BETWEEN REPAIRS

2.1 INTRODUCTION

This chapter deals with the optimality and the analysis of policy I by which a repair is carried out at failure or at a planned repair time, whichever occurs first. The emphasis is on the optimum schedules for maintenance or repairs, in order to minimize long-term average operating cost per unit time. This policy is defined in Section 2.2. A mathematical function representing the previous history of the system and an expression for long-term average cost per unit time are found in Section 2.3. The problem is formulated as nonlinear programming and dynamic programming in Sections 2.4 and 2.5, respectively. Existing results on state-dependent models are reviewed and a related improved computational algorithm is suggested in Section 2.6. Sections 2.7-2.9 are devoted to the age-dependent model and its optimality by means of a simulation technique and a suggested approximation procedure. The problem is formulated in Section 2.10 in terms of steady state availability.

2.2 POLICY I

Simple equipment such as a car engine may break down from time to time, needing repair to be restored to operation. Usually repair actions for a system having an increasing (or possibly nondecreasing)

failure rate cannot last forever, and at some point in time, replacement becomes inevitable. It is perhaps justifiable to perform the repairs prior to failures. To analyze this problem, a strategy called policy I has received the most attention in the literature. The following is a generalization of the type I policy.

Policy I. The system is repaired for the ith time in the case of failure or after T_i time units of continuing operation (from the last maintenance action), whichever occurs first ($i<N$). It is replaced when $i=N$. This process will be repeated indefinitely.

Using this policy contributes to the reduction of irregularity in breakdowns by performing planned repairs before the system fails. A discussion on this policy in a practical case will depend on considerations not all of which can be incorporated into a mathematical analysis. Therefore, the following general assumptions will be made.

1. Repairs are carried out instantaneously and there will be no queuing problem.

2. Information about the state of the system (failed, not failed) is obtained without any inspection. In other words, the system's state is observable.

3. *The failure time probability density function (p.d.f.) is known after each repair action and when the system is new.*

4. The system is subject to stochastic failures.

5. Distributions of failure times after each repair are not necessarily identical.

We will make additional assumptions as the parameters of this model are defined.

Several different criteria have been considered to determine the optimum planned repair times, among which minimizing the long-term average cost and maximizing the limiting efficiency (availability) are widely used. From the optimization point of view, these two criteria are actually equivalent (see Section 2.10).

We consider an optimum policy I for which the long-term average operating cost per unit of time is minimized. Each cycle, which is defined to be the time between two successive replacements, consists of N time periods (T_i, i=1,...,N), i.e. T_i is the time between the $(i-1)^{st}$ and i^{th} repair actions.

In this model the maintenance costs consist of replacement cost C_R, repair cost C_0, and breakdown cost C_B for each failure. These costs are assumed to be constant, even though, in practice, they might well be functions of age, number of repairs, and other external factors.

If C_0, C_B, and especially C_R are highly sensitive to the inflation rate and other economic factors, then long-term total discounted cost criterion seems to be preferable. In most situations, the costs can be interpreted as expected values of corresponding random variables.

C_0 represents all costs caused by performing a repair, including the cost associated with a constant repair time.

Breakdown cost C_B consists of all additional costs incurred due to failure. When planned repair is carried out, only C_0 is incurred.

It is also possible to define C_R, C_B, and C_0 as functions of time to adjust with depreciation or wear. But as we said earlier, it is assumed here that they are constant.

In the next section, a general type I policy is formulated.

2.3 POLICY I FORMULATION

A component or a system subject to stochastic failure can be described by its failure (hazard) rate $r(t)$

$$r(t)dt = Pr[t < \tau < t+dt | \tau > t] \tag{2.1}$$

where τ is a random variable specifying the failure time. The corresponding cumulative distribution function is

$$F(t) = 1 - \exp[-\int_0^t r(s)ds] \tag{2.2}$$

Let $\theta_i = g_i(T_1, T_2, \ldots, T_{i-1}; \tau_1, \tau_2, \ldots, \tau_{i-1})$ be a positive real function to describe the past aging of the system up to the $(i-1)^{st}$ preventive maintenance action. In this expression, T_i (constant planned repair time) and τ_i (failure time) are measured from the most recent repair or preventive maintenance. This allows us to formulate *the type I policy so that the failure rate is a function of age of the system* and/or number of repairs performed.

Failure characteristics of the system after the $(i-1)^{st}$ repair (or replacement if i=1) can now be described by the conditional cumulative failure distribution function $F_i(t|\theta_{i-1})$ and the corresponding conditional hazard rate $r_i(t|\theta_{i-1})$, which is assumed to be strictly increasing in $T_1, T_2, \ldots, T_N$, t, and also in the number of repairs (i-1). Assume that $F_i(t|\theta_{i-1})$ and $r_i(t|\theta_{i-1})$ are both continuous and differentiable for any positive value of the vector $\overline{T} = (T_1, T_2, \ldots, T_N)$ and t.

In order to have a meaningful preventive maintenance policy, we assume that after each repair, the system does not fully recover, and it is more likely to break down than if a replacement had taken place ("repaired not as good as new"). This is achieved by making the failure rate function after any repair satisfy

$$r_i(t|\theta_{i-1}) \geq r_{i-1}(t|\theta_{i-2}) \geq \ldots \geq r_1(t|\theta_0) \tag{2.3}$$
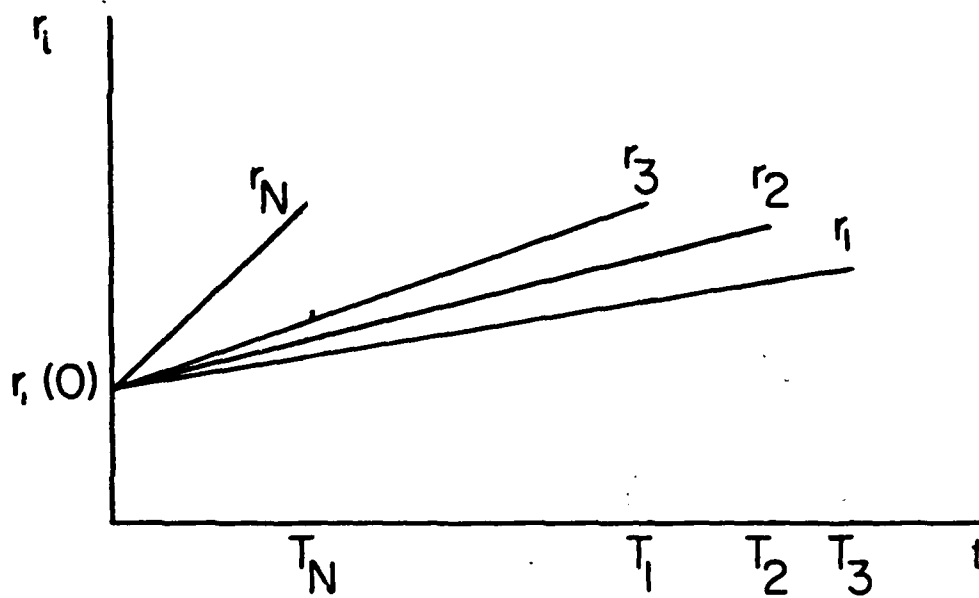
Figure 2.1- Various Linearly increasing Failure Rates in a cycle.

The function r_i might depend on i-1, $T_{i-1}, T_{i-2}, \ldots, T_1$. Figure 2.1 illustrates this fact when the failure rate is linearly increasing. Note that repairs will reduce the failure rates considerably such that

$$r_i(0|\theta_{i-1}) = r_{i-1}(0|\theta_{i-2}) = \ldots = r_1(0) \qquad (2.4)$$

During the ith period, the probability of a failure before the next preventive maintenance, given that (i-1) repairs have been performed, is expressed by $F_i(t_i|\theta_{i-1})$, and the expected breakdown cost for a cycle consisting of N period is

$$\text{Total Breakdown Cost} = C_B \sum_{i=1}^{N} F_i(T_i|\theta_{i-1}) \qquad (2.5)$$

The expected total cost per cycle denoted by $R(N, \overline{T})$ is found to be

$$R(N, \overline{T}) = C_R + (N-1)C_0 + C_B \sum_{i=1}^{N} F_i(T_i|\theta_{i-1}) \qquad (2.6)$$

where

$$\bar{T} = (T_1, T_2 \ldots, T_N)$$

To calculate the expected length of a cycle, we define the following:

τ_i = Random variable with p.d.f. $f(t|\theta_{i-1})$ denoting the failure time in period i, measured immediately after the $(i-1)^{st}$ repair (or replacement for i=1).

T_i = Scheduled repair time measured immediately after the $(i-1)^{st}$ repair (or replacement for i=1).

X_i = Age of period i, which is either T_i or τ_i, whichever occurs first.

Thus we have

$$X_i = \min(\tau_i, T_i) \tag{2.7}$$

using this notation, one possible age function is $\theta_i = 1 + \epsilon \sum_{j=1}^{i} X_j$; where >0 is an aging factor. The expected length of period i is computed by

$$y_i \triangleq E(X_i) = \int_0^{T_i} s f_i(s|\theta_{i-1}) ds + T_i \int_{T_i}^{\infty} f_i(s|\theta_{i-1}) ds \tag{2.8}$$

Making a change of variables

$$s = U \quad ; \quad f_i(s|\theta_{i-1}) ds = dV$$

or

$$ds = dU \quad ; \quad F_i(s|\theta_{i-1}) = V$$

and using integration by parts, we get

$$y_i = T_i F_i(T_i|\theta_{i-1}) - \int_0^{T_i} F_i(s|\theta_{i-1}) ds + T_i \bar{F}_i(T_i|\theta_{i-1})$$

or $\quad y_i = T_i[F_i(T_i|\theta_{i-1}) + \bar{F}_i(T_i|\theta_{i-1})] - \int_0^{T_i} F_i(s|\theta_{i-1})ds$

$$= T_i - \int_0^{T_i} F_i(s|\theta_{i-1})ds$$

$$= \int_0^{T_i} ds - \int_0^{T_i} F_i(s|\theta_{i-1})ds$$

and finally

$$y_i = \int_0^{T_i} \bar{F}_i(s|\theta_{i-1})ds \qquad (2.9)$$

Therefore, the expected length of a cycle denoted by $L(N,\bar{T})$ is expressed by

$$L(N,\bar{T}) \triangleq E(\sum_{i=1}^{N} X_i) = \sum_{i=1}^{N} \int_0^{T_i} \bar{F}_i(s|\theta_{i-1})ds \qquad (2.10)$$

According to the renewal reward theorem [Ross (1970)], the expected long-run average cost is the expected total cost during a cycle divided by the expected length of that cycle. Thus the expected long-term average cost per unit of time for policy I is

$$C(N,\bar{T}) = \frac{R(N,\bar{T})}{L(N,\bar{T})} = \frac{C_R + (N-1)C_0 + C_B \sum_{i=1}^{N} F_i(T_i|\theta_{i-1})}{\sum_{i=1}^{N} \int_0^{T_i} \bar{F}_i(t|\theta_{i-1})dt} \qquad (2.11)$$

In the next section, optimization of preventive maintenance policy I is given.

2.4 OPTIMAL POLICY I

Our goal is to find a set of positive time periods (T_i, $i = 1,2,\ldots,N$) so that the expected long-run average cost per unit time $C(N,\bar{T})$ is

minimized. This can be expressed by the following nonlinear programming problem (NPI):

Find N^* and $\overline{T}^* = (T_1(N), T_2(N), \ldots, T_N(N))$

to minimize

$$C(N, \overline{T}) = \frac{R(N, \overline{T})}{L(N, \overline{T})} \tag{2.12}$$

subject to

$$T_i \geq 0 \qquad i = 1, 2, \ldots, N \tag{2.13}$$

$$N \geq 1 \tag{2.14}$$

$$N \text{ integer} \tag{2.15}$$

First we consider three special cases:

Case I. No Repairs (N=1)

This case deals only with replacements, and repairs are not carried out. If N=1, the $C(N, \overline{T})$ will reduce to

$$C(1, T_1) = \frac{C_R + C_B F_1(T_1)}{\int_0^{T_1} \overline{F}_1(t)dt} \tag{2.16}$$

By setting its first derivative equal to zero, we obtain the first necessary condition for a minimum

$$C_B f_1(T_1)\int_0^{T_1} \overline{F}_1(t)dt - \overline{F}_1(T_1)[C_R + C_B F_1(T_1)] = 0 \tag{2.17}$$

Dividing both sides of (2.17) by $\overline{F}_1(T_1)$ and replacing $\dfrac{f_1(T_1)}{\overline{F}_1(T_1)}$ with $r_1(T_1)$ yields:

$$G_1(T_1) \triangleq r_1(T_1)\int_0^{T_1} \overline{F}_1(t)dt - F_1(T_1) = \frac{C_R}{C_B} \qquad (2.18)$$

An optimum type I policy is obtained by $T_1(1)$, which satisfies equation (2.18), and the corresponding minimum expected cost value is

$$C^*(1,T_1(1)) = C_B r_1(T_1(1)) \qquad (2.19)$$

Barlow and Proschan (1965) and Barlow and Hunter (1959) have considered a similar model in which they proved that optimal solution $T_1(1)$ may exist. Their discussion follows from the fact that the left-hand side of equation (2.18) is increasing in T_1 provided hazard rate $r_1(t)$ is increasing in t. Mathematically:

$$\frac{dG_1(T_1)}{dT_1} = r_1'(t_1)\int_0^{T_1} \overline{F}_1(t)dt + r_1(T_1)\overline{F}_1(T_1) - f_1(T_1)$$

$$= \overline{F}_1(T_1)\left[\frac{r_1'(T_1)\int_0^{T_1} F_1(t)dt}{\overline{F}_1(T_1)} + r_1(T_1) - \frac{f_1(T_1)}{\overline{F}_1(T_1)}\right]$$

$$= r_1'(T_1)\int_0^{T_1} \overline{F}_1(t)dt \geqq 0$$

The right-hand side of equation (2.18) is constant for any value of T_1. We conclude that $C(1,T)$ has at most one minimum point. But from (2.15)

$$C(1,T_1) \to \infty \quad \text{as} \quad T_1 \to 0$$

and

$$C(1,T_1) \to \frac{C_R + C_B}{\mu_1} \quad \text{as} \quad T_1 \to \infty$$

where μ_1 is the expected failure time. It was assumed previously that $C(1,T_1)$ is continuous. The two possibilities are shown in Figures 2.2(a) and 2.2(b).

In Figure 2.2(a), clearly, $T_1(1) = \infty$, but in 2.2(b), there exists a finite positive $T_1(1)$ for which $C(1,T_1(1)) < C(1,\infty)$.

Case II. Rapid Aging

In this case, the equipment deteriorates very rapidly and/or repairs are destructive. For any age function $\theta_i > 1$; $(i \geq 1)$ and $\theta_0 = 1$, the conditional failure rate may be defined as

$$r_i(t|\theta_{i-1}) = \theta_{i-1} r(t) \qquad (2.20)$$

from which the related conditional cumulative distribution function is expressed by

$$F_i(t|\theta_{i-1}) = 1 - \exp[-\theta_{i-1}\int_0^t r_i(s)ds]$$

$$= 1 - \exp[-\int_0^t r_i(s)ds]^{\theta_{i-1}} \qquad (2.21)$$

If θ_{i-1} $(i = 2,3,\ldots,N)$ is large, then

$$\lim_{\theta_{i-1} \to \infty} F_i(t|\theta_{i-1}) = 1 \qquad i = 2,3,\ldots,N$$

Clearly no planned repair must be scheduled, and the resulting expected long-term average cost per unit time given by (2.11) becomes

$$C(1,T) = \frac{C_R + C_B F_1(T_1)}{\int_0^{T_1} \overline{F}_1(t)dt}$$

which is the result of Case I.
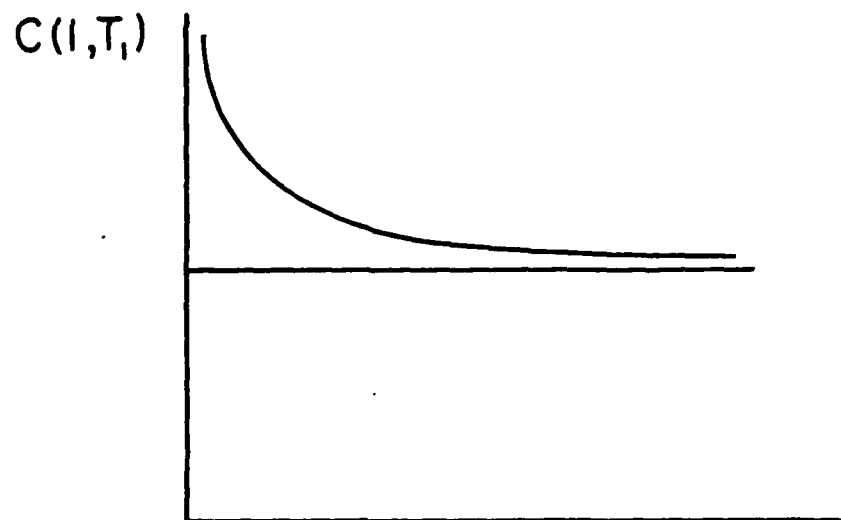
Figure 2.2 (a)
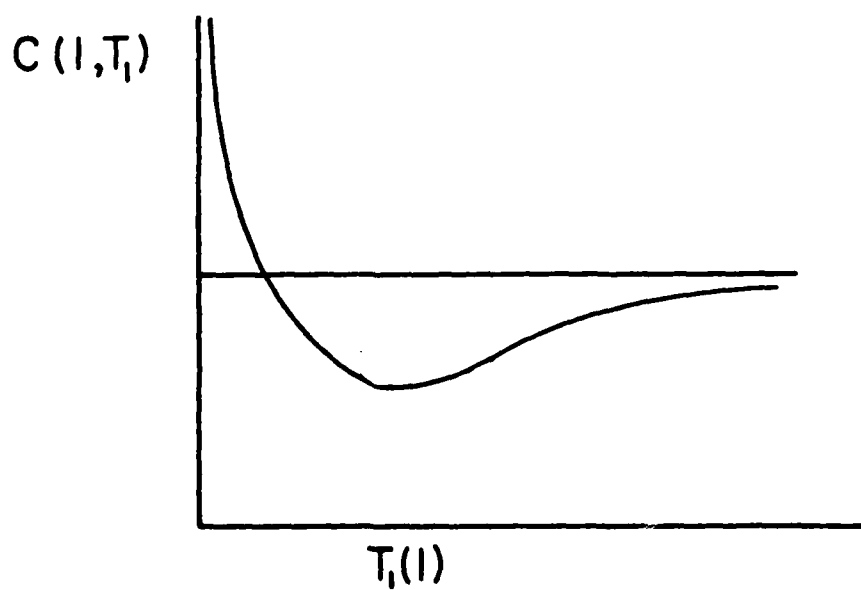
Figure 2.2 (b)

Case III. Slow aging

This situation deals with the case in which, after each repair, the system becomes as good as new. Mathematically, this is described by

$$r_i(t|\theta_{i-1}) = r(t) \tag{2.22}$$

Combining this relation and equation (2.11), we get

$$C(N,\overline{T}) = \frac{C_R + (N-1)C_0 + C_B \sum_{i=1}^{N} F(T_i)}{\sum_{i=1}^{N} \int_0^{T_i} \overline{F}(t)dt} \tag{2.23}$$

Taking its derivatives and equating them to zero gives

$$C_B f(T_j) \sum_{i=1}^{N} \int_0^{T_i} \overline{F}(t)dt - \overline{F}(T_j)[C_R + (N-1)C_0 + C_B \sum_{i=1}^{N} F(T_i)] = 0$$

$$\text{for } j = 1,2,\ldots,N \tag{2.24}$$

which can be simplified to

$$r(T_j) \sum_{i=1}^{N} \int_0^{T_i} \overline{F}(t)dt - \sum_{i=1}^{N} F(T_i) = \frac{C_R + (N-1)C_0}{C_B} \tag{2.25}$$

The minimum expected cost per unit of time is obtained by dividing (2.24) by $\overline{F}(T_j) \sum_{i=1}^{N} \int_0^{T_i} \overline{F}(t)dt$

$$C(N,\overline{T}^*) = C_B r(T_j(N)) \qquad \text{for all } j \text{ and } K \tag{2.26}$$

which means

$$r(T_j(N)) = r(T_K(N)) \qquad \text{for all } j \text{ and } K \tag{2.27}$$

Since the failure rate $r(t)$ is assumed to be an increasing function of t, we have

$$T_j(N) = T_K(N) = T \qquad \text{for } j \text{ and } K \tag{2.28}$$

And now equation (2.25) can be written as

$$Nr(T)\int_0^T \overline{F}(t)dt - NF(T) = \frac{C_R + (N-1)C_0}{C_B} \qquad (2.29)$$

Therefore, the optimum repair time for each period T is the solution to

$$G(T) \triangleq r(T)\int_0^T \overline{F}(t)dt - F(T) = \frac{C_R + (N-1)C_0}{NC_B} \triangleq h(N) \qquad (2.30)$$

The right-hand side of (2.30) is increasing in N if

$$\frac{C_R + (N-1)C_0}{NC_B} > \frac{C_R + NC_0}{(N+1)C_B} \qquad (2.31)$$

which implies

$$C_R > C_0 \qquad (2.32)$$

Obviously if $C_R < C_0$, then $h(N)$ is a decreasing function of N. Since the failure rate is increasing, the left-hand side of (2.30) is also increasing in T. To show this, we compute

$$\frac{dG(T)}{dT} = r'(T)\int_0^T \overline{F}(t)dt + r(T)\overline{F}(T) - f(T)$$

$$= r'(T)\int_0^T \overline{F}(t)dt + \frac{f(T)}{\overline{F}(T)} \cdot \overline{F}(T) - f(T)$$

$$= r'(T)\int_0^T \overline{F}(t)dt \geq 0 \qquad (2.33)$$

If replacement cost C_R is greater than repair cost C_0, then $h(N)$ decreases as we increase N, and consequently from (2.33)

$$T(N+1) < T(N) \qquad \text{for all N} \qquad (2.34)$$

which implies

$$C(N+1,\overline{T}) < C(N,\overline{T}) \qquad \text{for all } N \qquad (2.35)$$

[See Eq. (2.26)]

Therefore, if the replacement cost is more than repair cost, it is optimal not to replace the existing system as long as it is needed. The optimum repair time, $T(N)$, is the solution to

$$r(T)\int_0^T \overline{F}(t)dt - F(T) = \frac{C_o}{C_B} \qquad (2.36)$$

where $\dfrac{C_o}{C_B}$ is the limit of $h(N)$ as N approaches infinity.

If the replacement cost C_R is less than the repair cost C_o, then $h(N)$ is decreasing in N and

$$T(N+1) < T(N) \qquad \text{for all } N \qquad (2.37)$$

and from (2.26), we conclude

$$C(N+1,F) < C(N,\overline{T}) \qquad \text{for all } N \qquad (2.38)$$

Thus, the optimum number of scheduled repairs in this case is zero and only replacements are carried out. The procedure is again the same as for Case I ($N=1$).

Note that the breakdown cost C_B affects the above strategies by changing the length of $T(N)$ and, as a result, the value of $C(N,\overline{T})$, but it does not affect the optimal value of N. This is because a reduction in C_B will result in a larger $T(N)$. [$h(N)$ is decreasing in C_B.]

When $C_R = C_o$, at the time of failure or planned repair, we can either replace the system or perform a repair.

Case III, which is also a special case of a model developed by Nguyen and Murthy (1981), does not seem to be appropriate in prac-

tice, because the system's age and number of repairs are ignored. In order to have a nontrivial solution, we admit that the generalized policy I is applied when replacement cost C_R is greater than repair cost C_0.

Now we turn our attention to the general form of (NPI) and employ Dynamic Programming to reformulate it.

2.5 DYNAMIC PROGRAMMING FORMULATION

The objective is to select a set of positive repair times $\bar{T} = (T_1(N), T_2(N), \ldots, T_3(N))$ so that the expected average cost per unit time given by

$$C(N, \bar{T}) = \frac{R(N, \bar{T})}{L(N, \bar{T})} \tag{2.11}$$

is minimized where R and L have been defined by (2.6) and (2.10). It is well known [Barlow and Proschan (1965)] that $\bar{T}$, which minimizes $C(N, \bar{T})$ will also minimize a related problem given by

$$\mathcal{L}_1(\beta) = R(N, \bar{T}) - \beta L(N, \bar{T}) \tag{2.39}$$

Where $\beta^* = \beta$, the minimum of $C(N, \bar{T})$ is such that

$$\mathcal{L}_0^*(\beta) = 0 \tag{2.40}$$

Concentrating upon a single cycle, we define the following recursive equation

$$\mathcal{L}_K^*(\beta, \theta_{K-1}) = \underset{T_K}{\text{Min}} \{ C_K + \mathcal{L}_{K+1}^*(\beta, \theta_K) \} \tag{2.41}$$

C_K is the cost incurred during each period, and it is equal to

$$C_K = (1-\eta_K)C_R + \eta_K C_o + C_B F_K(T_K|\theta_{K-1}) - \beta \int_0^{T_K} \overline{F}_K(t|\theta_{K-1})dt$$

$$\text{for } K = 1,2,\ldots,N \qquad\qquad (2.42)$$

when

$$\eta_K = 1 \qquad \text{if} \qquad 1 \leq K < N$$

$$\eta_K = 0 \qquad \text{if} \qquad K = N$$

We now define θ_K as being proportional to the system's age at the time of the most recent repair. This can be written as a transition equation

$$\theta_K = \theta_{K-1} + \text{Min}(\tau_K, T_K) \qquad K \neq 0 \qquad (2.43)$$
$$\theta_o = 1$$

where $\in$ is a multiplier to determine the deterioration rate with respect to age. θ_K, which is a random variable, has the following distribution

$$\theta_K = \theta_{K-1} + \in T_K \qquad \text{with probability} \qquad \overline{F}(T_K|\theta_{K-1})$$

$$= \theta_{K-1} + \in \tau_K \qquad \text{with probability} \qquad F(T_K|\theta_{K-1}) \qquad (2.44)$$

The boundary conditions to (2.41) are

$$\mathcal{L}^*_{N+1}(\beta^*,\theta_N) = 0 \qquad\qquad (2.45)$$

and

$$\mathcal{L}_1(\beta,\theta_o) = 0 \qquad\qquad (2.46)$$

In most cases it seems unlikely to find an analytical solution for this problem. This is due to complexity caused by transition equation (2.43) and β. As an example, consider a system in which the failure

characteristic follows the Weibull distribution. In this situation, finding the distribution of θ_K requires numerical calculations. β^* is found at the last stage and its different values must be kept at each stage. It is most likely that carrying out these calculations will require a lot of execution time and storage space, and the results will lack precision.

2.6 STATE DEPENDENT MODEL

We define the state of the system to be the number of repairs performed. A mathematical representation of this model is

$$r_i(t|\theta_{i-1}) = r_i(t) \tag{2.46}$$

where i represents the number of repairs. This case describes a situation when the failure rate function increases with the number of repairs as well as time since the most recent repair, but the system's age at the time of repair does not play any significant role in its future performance. This problem has been studied by D.G. Nguyen and D.N.P. Murthy (1981).

From equation (2.11), the expected cost per unit time for this case is

$$C(N,\bar{T}) = \frac{C_R + (N-1)C_0 + C_B \sum_{i=1}^{N} F_i(T_i)}{\sum_{i=1}^{N} \int_0^{T_i} \bar{F}_i(t)dt} \tag{2.47}$$

As Nguyen and Murthy (1981) have shown by differentiating (2.47) with respect to the T_i (i = 1,2,...,N), the optimal preventive maintenance ages satisfy

$$r_i(T_i) = r_1(T_1) \quad ; \quad i = 2,3,...,N \tag{2.48}$$

and

$$\sum_{i=1}^{N} [r_1(t_1)\int_0^{T_i} \overline{F}_i(t)dt - F_i(T_i)] = \frac{(N-1)C_0 + C_R}{C_B} \qquad (2.49)$$

They have proved that under the following assumptions

i) $r_i(t)$ is strictly increasng to infinity (in t)

ii) $r_{i+1}(t) \geq r_i(t)$, t > 0

iii) $r_{i+1}(0) = r_i(0)$

the following are true:

1. For a fixed value of N, the optimal policy I exists and $T_i(N)$; (i = 1,2,...,N) is finite, unique and decreasing in i.

2. There exists an optimal N^* for which

$$\min_{N}[N|\Delta C(N,\overline{T} \geq 0] \leq N^* \leq \max_{N}[N|\Delta C(N,\overline{T}) \leq 0]$$

where $\Delta C(N,\overline{T}) = C(N+1,\overline{T}) - C(N,\overline{T})$

Nguyen and Murthy also proposed a computation algorithm as follows:

i) Set N = 1

ii) Obtain $T_i(N)$; i = 1,2,...,N using equations (2.48) and (2.49)

iii) If $T(N) \geq T(N-1)$, got to step (v)

iv) Set N = N+1 and go to step (ii)

v) $N^* = N-1$; compute $C(N^*,\overline{T})$ from equation (2.47)

(Note: step (iii) is omitted for N = 1.)

They have suggested that step (ii) be carried out by solving identities (2.48) and (2.49) simultaneously. However, applying any optimum seeking technique, such as gradient method, directly to equation

(2.47) also provides satisfactory results. Either technique seems to require numerical integration when underlying lifetime distributions are assumed to have the Weibull form. Consequently for a large N, the above algorithm is not likely to be efficient. For instance, if $N^* = 49$, then the last iteration of the algorithm alone requires $(50)(.2)(15)/60 = 2.5$ minutes of computer time for numerical integration (assuming that each numerical integration needs .2 seconds and step (ii) will converge after 15 iterations). In other words, to carry out this algorithm when $N^* = 49$, approximately one hour of computer time is spent to integrate numerically.

The following lemma will help us find an approximate value for N^*, which will be useful in later numerical calculations.

Lemma 2.1

For constant μ_i; $(i = 1,2,\ldots,n)$, a and b $(b > 0)$, the integer function

$$\rho_n = \frac{a + bn}{\sum_{i=1}^{n} \mu_i} \qquad n = 1,2,\ldots \qquad (2.50)$$

has at least one global minimum if

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_N \geq \ldots > 0 \qquad (2.51)$$

Proof. Let $\Delta\rho_n \triangleq \rho_{n+1} - \rho_n$ and from relation (2.50) we have

$$\Delta\rho_n = \frac{-a\mu_{N+1} + b(\sum_{i=1}^{n} \mu_i - n\mu_{n+1})}{(\sum_{i=1}^{n+1} \mu_i)(\sum_{i=1}^{n} \mu_i)} \qquad (2.52)$$

Note that if $a > 0$, and $\mu_{i+1} = \mu_i$ for all i, then $\Delta\rho_n < 0$, which implies that ρ_n decreases as n increases or $n^* = \infty$. If $a < 0$, then $\Delta\rho_n > 0$ for all n. Since $\sum_{i=1}^{n} \mu_i - n\mu_{n+1} > 0$ and $\mu_i \geq \mu_{n+1}$; $i = 1,2,\ldots,n$ (from 2.52) and consequently $n^* = 1$. For $a > 0$ and $b > 0$, first we will show that if $\Delta\rho_n > 0$, then $\Delta\rho_{n+1} > 0$. Identity (2.52) is positive when

$$-a\mu_{n+1} + b(\sum_{i=1}^{n} \mu_i - n\mu_{n+1}) > 0 \tag{2.53}$$

or

$$\frac{\sum_{i=1}^{n} \mu_i - n\mu_{n+1}}{\mu_{n+1}} > \frac{a}{b} \tag{2.54}$$

Similarly $\Delta\rho_{N+1} > 0$ if and only if

$$\frac{\sum_{i=1}^{n+1} \mu_i - (n+1)\mu_{n+2}}{\mu_{n+2}} > \frac{a}{b} \tag{2.55}$$

which can also be written as

$$\frac{\sum_{i=1}^{n} \mu_i - n\mu_{n+2} + \mu_{n+1} - \mu_{n+2}}{\mu_{n+2}} > \frac{a}{b} \tag{2.56}$$

We can increase the left-hand side of inequality (2.54) via replacing μ_{n+1} by μ_{n+2} and adding $\mu_{n+1} - \mu_{n+2} > 0$ to the numerator. This will result in inequality (2.56) and, hence we conclude that $\Delta\rho_n > 0$ implies $\Delta\rho_{n+1} > 0$.

Now we have to prove that $\Delta\rho_{n+1} < 0$ implies $\Delta\rho_n < 0$. For $\Delta\rho_{n+1} < 0$, it can be shown that

$$\frac{\sum\limits_{i=1}^{n} \mu_i - n\mu_{n+2} + \mu_{n+1} - \mu_{n+2}}{\mu_{n+2}} < \frac{a}{b} \qquad (2.57)$$

The inequality holds if we decrease the left-hand side. Replacing μ_{n+2} by μ_{n+1} yields

$$\frac{\sum\limits_{i=1}^{n} \mu_i - n\mu_{n+1}}{\mu_{n+1}} < \frac{a}{b} \qquad (2.58)$$

which means $\Delta\rho_n < 0$ and the proof is completed.

One may define the following policy:

<u>Policy I'</u>: The system is repaired for the ith time in the case of failure if $i < N'$. It is replaced when $i = N'$.

In fact this strategy is the same as policy I, but planned repair times have been selected so large that failures occur prior to the scheduled preventive maintenance. The long-term average cost per unit time is obtained by letting $T_i \to \infty$ $(i = 1,2,\ldots,N)$ in identity (2.11)

$$C(N',\infty) = \frac{C_R - C_o + (C_o + C_B)N'}{\sum\limits_{i=1}^{N'} \mu_i} \qquad (2.59)$$

where μ_i is the expected failure time in the ith period. It has been shown by Nguyen and Murthy that for a state-dependent model with the assumed failure rate function properties, $\mu_i \geq \mu_{i+1} > 0$; $i = 1,2,\ldots$. In fact, this attribute exists in age and state-age-dependent models. This follows from the fact that whenever $r_i(t|\theta_{i-1})$ is less than $r_{i+1}(t|\theta_i)$, one can see easily that $\overline{F}_i(t|\theta_{i-1}) < \overline{F}_i(t|\theta_{i-1})$ and this leads to $\mu_{i+1} < \mu_i$.

Setting $C_R - C_0 = a$, $C_0 + C_B = b$, and $N' = n$, we get equation (2.50) given in lemma 2.1, and hence we conclude that optimal policy I' exists. There is a unique N' which minimizes the expected cost per unit time (2.59) except the case in which two or more values of N' will result in the same amount of cost $C(N', \infty)$. In other words, function (2.59) is either convex or strictly convex function of N'. It is important to note that the behavior of $C(N, \infty)$, when $\mu_{i+1} = \mu_i$ (policy I'), and special case III (policy I) are exactly the same. In both models, $N^* = N'^* = \infty$, when $C_R > C_0$, and for $C_R < C_0$, no repair should be scheduled $(N^* = N'^* = 1)$.

Calculation of N'^* is simple. T_i's have been eliminated, and there is no numerical integration to evaluate. In most cases, it can be done by hand. But the question is how well N^* is approximated by N'^*.

Our simulation study shows that N' is a good estimate for N, and due to the lack of preventive maintenance actions, the average cost for strategy I' is higher than policy I; consequently, $N'^* \leq N^*$. This allows us to use N'^* as a starting value for N. Employing this heuristic procedure will save a large amount of computer time by avoiding the computation used to evaluate the iterations $1, 2, \ldots, N'-1$.

Numerical Example 2.1

Referring to the numerical example given by Nguyen and Murthy where $C_0 = 5$, $C_R = 15$, $C_B = 15$, and failure times follow the Weibull distributions, i.e.

$$f_i(t) = \lambda_i t^\alpha \cdot \exp(-\frac{\lambda_i t^{\alpha+1}}{\alpha+1}) \qquad i = 1, 2, \ldots, N \quad .$$

with constant shape parameter $\alpha = 1$ and scale parameters $\lambda_1 = 1$, $\lambda_i = (1.5)^{i-1}$; $i = 2,3,\ldots,N$. The optimum type I policy is obtained by $T_1(3) = 0.936$, $T_2(3) = 0.624$, $T_3(3) = 0.416$, $N^* = 3$ and $C(N,\overline{T}^*) = 28.08$. But our heuristic procedure, policy I', yields $N'^* = 2$. By selecting initially $N = 2$, we could have avoided more complicated computations required for $N = 1$. The results are shown in Table 2.1.

N'	C(N',∞)
1	33.85
2	31.06
3	31.80

Table 2.1. Optimal Policy I' ($N'^* = 2$)

Numerical Example 2.2

This example illustrates a case for which calculations of N' can further be simplified. Suppose each failure (and consequently repair) affects the expected lifetime of the following period such that

$$\mu_i = \left(\frac{n^2-i+1}{n^2}\right)\mu_0 = \left(1 - \frac{i-1}{n^2}\right)\mu_0 \qquad i = 1,2,\ldots,n \qquad (2.60)$$

where $\mu_1 = \mu_0$. Note that μ_i is decreasing in i and the expected length of the ith time interval is less than the expected $(i-1)^{st}$ lifetime.

The equation (2.50), for this example, becomes

$$\rho_n = \frac{a + bn}{\sum\limits_{i=1}^{n} \left(1 - \frac{i-1}{n^2}\right)\mu_0} = \frac{a + bn}{\mu_0\left(n - \frac{n-1}{2n}\right)}$$

We may treat n as a continuous variable and differentiate ρ_n with respect to n. Setting that derivative to zero yields

$$(2a + b)n^2 - 2bn - a = 0 \qquad (2.61)$$

The positive solution to this quadratic equation is

$$n = \frac{b = \sqrt{b^2 + ab + 2a^2}}{2a + b} \tag{2.62}$$

Setting $a = C_R - C_o$ and $b = (C_o + C_B)$, we get

$$n = \frac{C_o + C_B + \sqrt{2C_o^2 + C_R(2C_R + 3C_o) + C_B(C_R + C_o + C_B)}}{2C_R - C_o + C_B} \tag{2.63}$$

The optimum N' is either [n] or [n]+1, depending on the value of $C(N', \infty)$. [n] denotes the largest integer less than or equal to n.

2.7 AGE-DEPENDENT MODEL

We are now considering a model that is applied to situations in which repairs do not reflect any increase in the system's failure rate. But, due to the length of operating time, recovery is not fully achieved by performing a repair. According to our notation, θ_i becomes an age factor that relates the system's age at the end of the ith period to its future performance. For simplicity, we assume that the conditional hazard rate for each period is proportional to the age factor θ_{i-1} at the beginning of that period

$$r_i(t | \theta_{i-1}) = r(t)\theta_{i-1} \tag{2.64}$$

This implies that

$$\bar{F}(t | \theta_{i-1}) = [\bar{F}(t)]^{\theta_{i-1}} \tag{2.65}$$

and θ_{i-1} is restricted to

$$\begin{aligned} \theta_{i-1} &> 1 \quad , \quad \text{if } i > 1 \\ &= 1 \quad , \quad \text{if } i = 1 \end{aligned} \tag{2.66}$$

In fact, θ_i is a random variable defined by

$$\theta_i = g_i(X_1 X_2, \ldots, X_{i-1}) \tag{2.67}$$

where g_i is an increasing function of X_1, $X_2, \ldots, X_{i-1}$ (X_i is given by equation 2.7). An appropriate example of g_i is

$$\theta_i = 1 + \epsilon \sum_{j=1}^{i} X_j \tag{2.68}$$

which satisfies the conditions given by (2.66) and 2.67). ϵ is a positive parameter which determines how fast the system deteriorates with respect to its age. A simulation study seems to be appropriate to analyze this age-dependent type I policy. We may also approximate relation (2.68) by

$$\theta_i = 1 + \epsilon \sum_{j=1}^{i} y_j \tag{2.69}$$

where

$$y_j \triangleq E(X_j) \tag{2.70}$$

This way, the failure rates will depend on the expected value of age as a constant rather than age as a random variable.

In the next two sections, the age-dependent policy I is formulated in two ways, one via a simulation approach and the other by approximation, using equations (2.69) and (2.70) instead of (2.68).

2.8 SIMULATION APPROACH

A key step in the analysis of the age-dependent type I policy via simulation is to find expressions for total cost per cycle, $\underset{\sim}{R}(N,\bar{T})$, and

length of a cycle $\underset{\sim}{L}(N,\overline{T})$, in terms of random variables involved. For this to be the case, let

$$\delta_i = \begin{cases} 1, & \text{if failure time} < T_i \\ 0, & \text{otherwise} \qquad\qquad i = 1,2,\ldots,N \end{cases} \tag{2.71}$$

This is equivalent to

$$\Pr\{\delta_i=1\} = F_i(T_i|\theta_{i-1}) \tag{2.72}$$

and

$$\Pr\{\delta_i=0\} = 1-F_i(T_i|\theta_{i-1}) \tag{2.73}$$

According to policy I, breakdown cost C_B is incurred if the system fails to operate before the scheduled repair time. Given this notion, and using the definition given for δ_i (2.71), the total cost per cycle, $\underset{\sim}{R}(N,\overline{T})$ (random variable), is expressed by

$$\underset{\sim}{R}(N,\overline{T}) = C_R + (N-1)C_0 + C_B\sum_{i=1}^{N}\delta_i \tag{2.74}$$

A random variable representing the length of a cycle is

$$\underset{\sim}{L}(N,\overline{T}) = \sum_{i=1}^{N} X_i$$

$$= \sum_{i=1}^{N} \min(\tau_i,T_i) \tag{2.75}$$

or by employing δ_i, it can be shown as

$$\underset{\sim}{L}(N,\overline{T}) = \sum_{i=1}^{N} [\delta_i\tau_i + (1-\delta_i)T_i] \tag{2.76}$$

Let $\underset{\sim}{R}^{(i)}(N,\bar{T})$ and $\underset{\sim}{L}^{(i)}(N,\bar{T})$ be the result of the ith simulation experiment and let us assume that n such experiments are to be performed, where n is a fairly large number $(50 \leq n \leq 100)$. Then the long-term average cost per unit time consistent with relation (2.11) is obtained by

$$C(N,\bar{T}) = \frac{\sum\limits_{i=1}^{n} \underset{\sim}{R}^{(i)}(N,\bar{T})}{\sum\limits_{i=1}^{n} \underset{\sim}{L}^{(i)}(N,\bar{T})} \qquad (2.77)$$

As one might expect, the above formulation is a general representation of type I policy, and it is applied to state-, age-, or state-age-dependent models. But generating a random variable, τ_i, differs for each case.

Among the most often used random number generators is the multiplicative, congruential method. But as Marsaglia (1968) has pointed out, this technique provides unsatisfactory results when sequences of random vectors are to be generated. If K-tuples $(RN_1, RN_2, \ldots, RN_K)$, $(RH_2, RN_3, \ldots RN_{K+1}), \ldots$ of random numbers generated by this method are viewed as points of a unit cube (K dimensions), all the points will lie in a relatively small number of hyperplanes. This property indicates that multiplicative congruential method is inadequate for our type I policy. Marsaglia and Bray (1968) have suggested the use of a composite generator, which mixes three congruential generators: one to fill n_1 locations, one to select a location from the n_1, and the third is used for good measure.

The following algorithm can be used to generate n random variables having the distribution $F(t|\theta_{i-1}) = 1-[\bar{F}(t)]^{\theta_{i-1}}$, which is appropriate for the age-dependent type I policy.

Step 0:   Set $\theta_0 = 1$; $i = 0$

Step 1:   Set $i = i+1$

Step 1:   Generate τ_i, having distribution $F(t|\theta_{i-1}) = 1 - [\overline{F}(t)]^{\theta_{i-1}}$,
using random number RN_i

Step 3:   If $i = N$, stop; else go to Step 4

Step 4:   Compute $X_i = \min(\tau_i, T_i)$

Step 5:   Compute $\theta_i = \theta_{i-1} + \in X_i$

Step 6:   Go to Step 1

As an example, consider a model in which the underlying distribution is Weibull, i.e.

$$f_i(t|\theta_{i-1}) = \lambda_o \theta_{i-1} t^\alpha \exp(- \frac{\lambda_o \theta_{i-1} t^{\alpha+1}}{\alpha+1}) \qquad i = 1,2,\ldots,N \qquad (2.78)$$

where

$$\theta_{i-1} = 1 + \sum_{j=1}^{i-1} \min(\tau_i, T_i) \qquad (2.79)$$

and $\theta_0 = 1$. Step 2 is performed by

$$\tau_i = \frac{-(\alpha+1)\ln(1-RN_i)}{\lambda_o \theta_{i-1}}^{\frac{1}{\alpha+1}} \qquad (2.80)$$

Calculation of δ_i can be done by either looking at $\mathrm{Min}(\tau_i, T_i)$ or if

$$RN_i < F(T_i|\theta_{i-1}) \qquad (2.81)$$

Then $\delta_i = 1$, otherwise $\delta_i = 0$.

Another problem that can play a significant role in finding the optimal policy I is selecting T_1, $T_2,\ldots,T_N$. Obviously, simulating all possible values will lead to inefficiency. However, it is reasonable to

say that the maximum length of period i (a good system) should be longer than the one of period (i+1) (not as good a system). Our simulation study also proves this fact when the lifetime distribution is Weibull. Given this notion, it is reasonable to reduce the dimensionality of the search by assuming a proportional relationship among the T_i. We may define $T_{i+1} = aT_i$; (i = 1,2,...,N-1; 0<a<1), and then find the optimal T_1, N, and a. This method is quite simple, but for a large N and a<<1, there is a K such that for i>K, the value of T_i is no longer significant. However, it can also be shown by numerical examples (e.g. Weibull) that there is not a constant proportionality among optimal times $T_1, T_2, ..., T_N$. In the Weibull case, we obtained better results by letting the reliability of the system in period i+1 be related to the reliability of period i by a constant value B, i.e.

$$Pr\{t > T_{i+1}\} = [Pr\{t > T_i\}]^B \qquad (2.82)$$

$$\text{or } e^{-\int_0^{T_{i+1}} r_{i+1}(t|\theta_i) dt} = e^{-B\int_0^{T_i} r_i(t|\theta_{i-1}) dt} \qquad (2.83)$$

This becomes

$$\int_0^{T_{i+1}} r_{i+1}(t|\theta_i)dt = B\int_0^{T_i} r_i(t|\theta_{i-1})dt \qquad (2.84)$$

For a Weibull distribution with hazard rate $r_i(t|\theta_{i-1}) = \lambda_o \theta_{i-1} t^\alpha$, identity (2.84) leads to

$$\frac{\lambda_o}{\alpha+1} \theta_i T_{i+1}^{\alpha+1} = \frac{B\lambda_o}{\alpha+1} \theta_{i-1} T_i^{\alpha+1} \qquad (2.85)$$

or

$$T_{i+1} = \left(\frac{B\theta_{i-1}}{\theta_i}\right)^{\frac{1}{1+\alpha}} T_i = A\left(\frac{\theta_{i-1}}{\theta_i}\right)^{\frac{1}{\alpha+1}} T_i \qquad (2.86)$$

where $A = B^{\frac{1}{\alpha+1}}$. In most cases, the optimal A seems to be close to one $(.95 \leqq A \leqq 1)$.

Numerical Example 2.3

Assume failure times of a system follow the Weibull distribution given by (2.78) and (2.79) with $\lambda_0 = 1$, $\alpha = 1$, and $\epsilon = .2$. The maintenance costs are $C_R = 15$, $C_0 = 5$, and $C_B = 12$. The optimal solution obtained by employing the simulation approach (by assuming that the relation 2.86 satisfying among the T_i) is $N^* = 7$, $A^* = 1$, $\overline{T}^* = (0.9 \quad 0.83 \quad 0.78 \quad 0.73 \quad 0.71 \quad 0.68 \quad 0.67)$ and $C(7,\overline{T}^*) = 15.08$ where $C(7,\infty) = 18.23$. Figure 2.3 shows the behavior of $C(N,\overline{T}^*)$ as N is incremented. The result is a convex function, and it has a global minimum.

In a manner similar to the state dependent model in Section 2.5, the computations may be reduced by finding an estimate for N. This is achieved by simulating a similar but not as good policy, namely letting $\overline{T} \to \infty$ $(\delta_i = 1)$. Equation (2.74) becomes

$$\underset{\sim}{R}(N,\overline{T}) = C_R + (N-1)C_0 + NC_B \qquad (2.87)$$

and equation (2.75) will change to

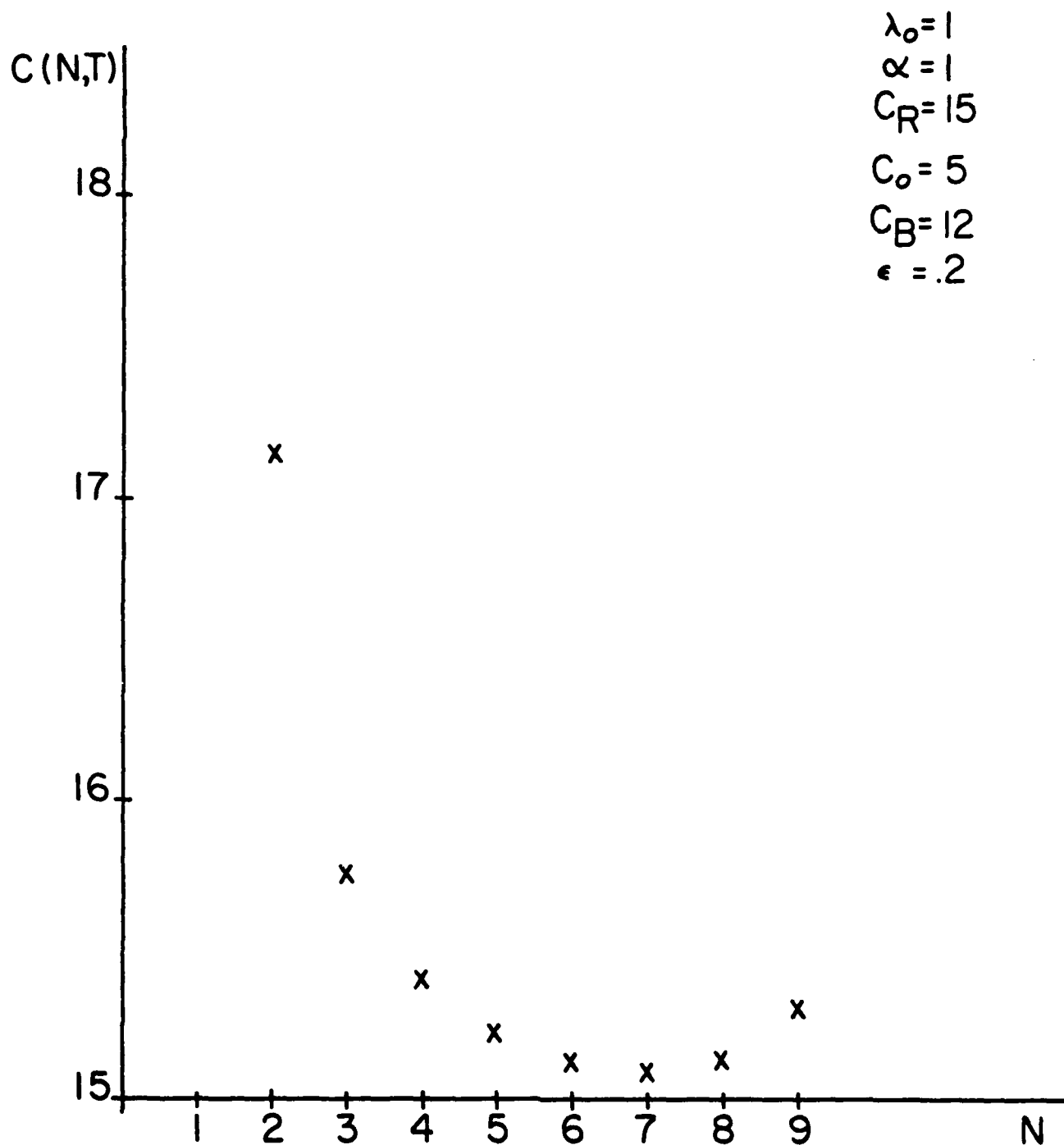$$\underset{\sim}{L}(N,\overline{T}) = \sum_{i=1}^{N} \tau_i \qquad (2.88)$$

Figure 2.3 – optimal Policy I using simulation
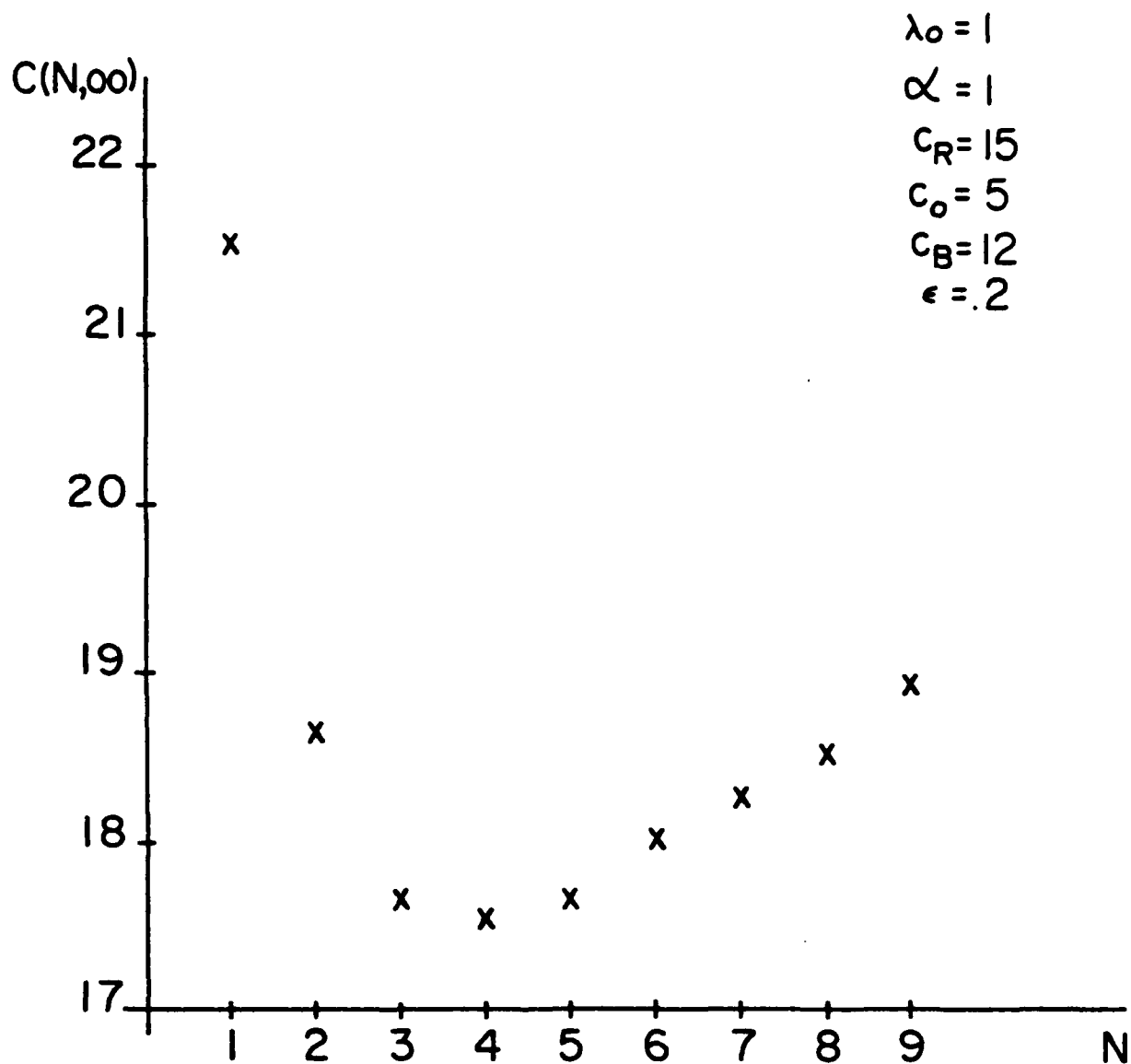
Figure 2.4 - The behavior of C(N,OO)

because of the higher cost in this case. The optimal N (say N'^*) is less than the one of actual policy I, and N' may be used as a starting value for N. Figure 2.4 illustrates the behavior of $C(N,\infty)$ as N is incremented (for the numerical example 2.3). It has also a global minimum. The estimated number of scheduled repairs for this case is $N'^* = 4$. By starting from N = 4, computations required for N = 1, 2, and 3 could have been replaced by fewer computations needed to obtain N'^*.

2.9 APPROXIMATION

Although simulation provides a flexible and significant approach to the age-dependent type I policy, nevertheless, one may wish to have rather an approximate analytical formulation to this problem. For this to be the case, we let

$$\theta_i = 1 + \sum_{j=1}^{i} E[\min(\tau_j, T_j)] = 1 + \epsilon \sum_{j=1}^{i} y_j \qquad (2.89)$$

where y_i is the expected age of the period defined by

$$y_i = \int_0^{T_i} F(t|\theta_{i-1})dt \qquad (2.90)$$

In other words, the failure rate at the beginning of each stage depends on the expected current age rather than the actual age. Therefore, the expected cost per unit time (2.11) can be written as

$$C(N,\overline{T}) = \frac{R(N,\overline{T})}{L(N,\overline{T})} = \frac{C_R + (N-1)C_0 + C_B \sum_{i=1}^{N} F(T_i|\theta_{i-1})}{\sum_{i=1}^{N} y_i} \qquad (2.91)$$

Since $\frac{\partial y_i}{\partial T_i} = \overline{F}(T_i|\theta_{i-1})$, (2.91) can be rewritten as

$$C(N,\overline{T}) = \frac{C_R + (N-1)C_0 + C_B \sum_{i=1}^{N}(1-\frac{\partial y_i}{\partial T_i})}{\sum_{i=1}^{N} y_i} \qquad (2.92)$$

According to our assumption $r(t|\theta_{i-1}) = r(t)\theta_{i-1}$ and hence

$$\overline{F}(t|\theta_{i-1}) = \exp[-\int_0^{T_i} r(t)\theta_{i-1}dt]$$

$$= [\exp(-\int_0^{T_i} r(t)dt)]^{\theta_{i-1}}$$

$$= [\overline{F}(t)]^{\theta_{i-1}} \qquad (2.93)$$

or

$$\overline{F}(t|\theta_{i-1}) = \left[\overline{F}(t) \right]^{1+\epsilon\sum_{j=1}^{i-1} y_j} \qquad (2.94)$$

which is an indication of reduction in reliability as the expected age increases and ϵ will determine the magnitude of that reduction.

To find an expression satisfying the necessary condition of the optimality, denote the first partial derivatives of relation (2.92) by

$$C_j(N,\overline{T}) = \frac{\partial C(N,\overline{T})}{\partial T_j} \qquad j = 1,2,\ldots,N \qquad (2.95)$$

and setting $C_J(N,\overline{T})$ to zero yields

$$-C_B \sum_{i=j}^{N} \frac{\partial^2 y_i}{\partial T_j \partial T_i} \sum_{i=1}^{N} y_i - \sum_{i=j}^{N} \frac{\partial y_i}{\partial T_j}\left[C_R + (N-1)C_0 + C_B \sum_{i=1}^{N}(1-\frac{\partial T_i}{\partial T_i}) \right] = 0$$

$$(2.96)$$

which can be simplified to

$$- \sum_{i=j}^{N} \frac{\partial^2 y_i}{\partial T_j \partial T_i} \bigg/ \sum_{i=j}^{N} \frac{\partial y_i}{\partial T_j} \quad \sum_{i=1}^{N} y_i - \sum_{i=1}^{N} (1 \frac{\partial y_i}{\partial T_i}) = \frac{C_R + (N-1)C_o}{C_B} \qquad (2.97)$$

$$j = 1,2,\ldots,N$$

Identity (2.96) can also be written as

$$\frac{C_R + (N-1)C_o + C_B \sum_{i=1}^{N} (1 - \frac{\partial y_i}{\partial T_i})}{\sum_{i=1}^{N} y_i} = -C_B \sum_{i=j}^{N} \frac{\partial^2 y_i}{\partial T_j \partial T_i} \bigg/ \sum_{i=j}^{N} \frac{\partial y_i}{\partial T_j} \qquad (2.98)$$

$$(j = 1,2,\ldots,N)$$

The left-hand side is just the minimum expected cost per unit time. When j=N from equation (2.90), we have

$$\frac{\partial y_N}{\partial T_N} = \overline{F}(T_N | \theta_{N-1}) \qquad (2.99)$$

and

$$\frac{\partial^2 y_N}{\partial T_N^2} = -f(T_N | \theta_{N-1}) \qquad (2.100)$$

hence the right-hand side of (2.98) is

$$RHS = \frac{C_B f(T_N | \theta_{N-1})}{\overline{F}(T_N | \theta_{N-1})} = C_B r(T_N | \theta_{N-1}) \qquad (2.101)$$

Therefore, the minimum expected cost per unit time can be expressed by

$$C(N, \overline{T}^*) = C_B r(T_N | \theta_{N-1}) \qquad (2.102)$$

The above computations can also be used to simplify one of the equations (j=N) given by (2.97) to get

$$r(T_N|\theta_{N-1}) \sum_{i=1}^{N} y_i - \sum_{i=1}^{N} F(T_N|\theta_{N-1}) = \frac{C_R + (N-1)C_o}{C_B} \qquad (2.103)$$

Evaluating (2.97 for j=K and K+1 and subtracting the results, we get another system of equations for the first necessary condition of the optimality.

$$\left(\sum_{i=K+1}^{N} \frac{\partial^2 y_i}{\partial T_{K+1} \partial T_i} \Big/ \sum_{i=1}^{N} \frac{\partial y_i}{\partial T_{K+1}} \right) = \left(\sum_{i=K}^{N} \frac{\partial^2 y_i}{\partial T_K \partial T_i} \Big/ \sum_{i=K}^{N} \frac{\partial y_i}{\partial T_K} \right)$$

$$K = 1,2,\ldots,N-1 \qquad (2.104)$$

Despite the fact that equations (2.97) or (2.104) along with (2.103) provide a way to obtain the optimal solution, they seem to be quite tedious and difficult to solve. One might prefer to apply an optimum search technique (e.g., conjugate gradient) directly to identity (2.91) and have a more convenient and efficient method for the evaluation of optimal age-dependent policy I. For this to be the case, we must find expressions for gradients $C_j(N,\overline{T})$, $j = 1,2,\ldots,N$. This in return requires the derivations of

$$R_j(N,\overline{T}) = \frac{\partial T(N,\overline{T})}{\partial T_j} \qquad\qquad j = 1,2,\ldots,N \qquad (2.105)$$

and

$$L_j(N,\overline{T}) = \frac{\partial L(N,\overline{T})}{\partial T_j} \qquad\qquad j = 1,2,\ldots,N \qquad (2.106)$$

To find $R_j(N,\overline{T})$, it is simplest to proceed by induction. This will result in

$$R_j(N,\bar{T}) = C_B\left[(1+\epsilon\sum_{i=1}^{j-1}y_i)f(T_j)\left[\bar{F}(T_j)\right]^{\epsilon\sum_{i=1}^{j-1}y_i}\right.$$

$$\left. - \epsilon\sum_{m=j+1}^{N}(\sum_{i=j}^{m-1}\frac{\partial y_i}{\partial T_j})\left[\bar{F}(T_m)\right]^{1+\epsilon\sum_{i=1}^{m-1}y_i}\ln\bar{F}(T_m)\right]$$

$$(\sum_{m=j+1}^{N}(\ ) = 0 \text{ when } j = N) \qquad\qquad j = 1,2,\ldots,N \qquad (2.107)$$

y_j is the first term in $L(N,\bar{T})$ which involves T_j. Hence

$$L_j(N,\bar{T}) = \sum_{i=j}^{N}\frac{\partial y_i}{\partial T_j} \qquad\qquad j = 1,2,\ldots,N \qquad (2.108)$$

The partial definative of y_K with respect to T_i is obtained from relation (2.90)

$$\frac{\partial y_K}{\partial T_i} = \epsilon(\sum_{j=i}^{K-1}\frac{\partial y_j}{\partial T_i})\int_0^{T_K}\left[\bar{F}(t)\right]^{1+\epsilon\sum_{j=1}^{K-1}y_j}\ln\bar{F}(t)dt \qquad \text{for } i = 1,2,\ldots,K-1$$

$$= \left[\bar{F}(T_i)\right]^{1+\epsilon\sum_{j=1}^{i-1}y_j} \qquad\qquad \text{for } i = K$$

$$= 0 \qquad\qquad \text{for } i = K,K+1,\ldots,N$$

$$K = 1,2,\ldots,N \qquad\qquad (2.109)$$

In order to represent the above formulas in a rather simpler fashion, let

$$d_{Ki} = \frac{\partial y_K}{\partial T_i} \qquad\qquad i = 1,2,\ldots N \qquad (2.110)$$

and also let

$$V_K = -\int_0^{T_K}\left[\overline{F}(t)\right]^{1+\epsilon\sum_{j=1}^{K-1}y_j}\ln\overline{F}(t)dt \qquad (2.111)$$

$$K = 1,2,\ldots,N$$

Since $\ln\overline{F}(t) \leq 0$, then V_K will be positive. Thus (2.109) can be re-written as

$$d_{Ki} = -\epsilon(\sum_{j=i}^{K-1}d_{ji})V_K \qquad \text{for } i = 1,2,\ldots,K-1$$

$$= \left[\overline{F}(T_i)\right]^{1+\epsilon\sum_{j=1}^{i-1}y_j} \qquad \text{for } i = K$$

$$= 0 \qquad \text{for } i = K+1,K=2,\ldots,N \qquad (2.112)$$

As one might expect, we can now form a matrix d, whose elements d_{Ki} are related by relation (2.112). This matrix is used for calculations of both $R_i(N,\overline{T})$ and $L_i(N,\overline{T})$.

Matrix d is shown by

$$\begin{matrix} d_{11} & 0 & 0 & \ldots & 0 \\ d_{21} & d_{22} & 0 & \ldots & 0 \\ d_{31} & d_{32} & d_{33} & \ldots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ d_{N1} & d_{N2} & d_{N3} & \ldots & d_{NN} \end{matrix} \qquad (2.113)$$

Equation (2.108) can be calculated by using matrix d

$$L_i(N,\overline{T}) = \sum_{j=i}^{N}d_{ji}$$

$$= \text{sum of column } i \qquad (2.114)$$

Calculation of $R_i(N,\overline{T})$ also requires partial summation along each column of matrix d. After obtaining $R_i(N,\overline{T})$ and $L_i(N,\overline{T})$, the gradients are simply expressed by

$$C_i(N,\overline{T}) = \frac{R_i(N,\overline{T})L(N,\overline{T}) - L_i(N,\overline{T})R(N,\overline{T})}{[L(N,\overline{T})]^2} \tag{2.115}$$

As an example, let us assume that the underlying lifetime distribution is Weibull given by

$$f_i(t|\theta_{i-1}) = \lambda_0\theta_{i-1}t^{\alpha}\exp(-\frac{\lambda_0\theta_{i-1}t^{\alpha+1}}{\alpha+1}) \tag{2.116}$$

$$\alpha > 1 \quad ; \quad \lambda_0, t > 0$$

where

$$\theta_{i-1} = 1 + \epsilon\sum_{j=1}^{i-1}y_j \tag{2.117}$$

From identity (2.107), $R_i(N,\overline{T})$ for this case is

$$R_i(N,\overline{T}) = C_B\left[\lambda_0\theta_{i-1}T_i^{\alpha}\exp(\frac{-\lambda_0\theta_{i-1}T_i^{\alpha+1}}{\alpha+1})\right.$$

$$\left. + \frac{\epsilon\lambda_0}{\alpha+1}\sum_{m=i+1}^{N}(\sum_{j=i}^{m-1}d_{ji})T_m^{\alpha+1}\exp\frac{-\lambda_0\theta_{i-1}T_m^{\alpha+1}}{\alpha+1}\right]$$

$$i = 1,2,\ldots,N \tag{2.118}$$

and from (2.111), V_i is calculated to be

$$V_i = \frac{\lambda_0}{\alpha+1}\int_0^{T_i}\exp(-\frac{\lambda_0\theta_{i-1}t^{\alpha+1}}{\alpha+1})t^{\alpha+1}dt \tag{2.119}$$

Let $\quad \dfrac{\lambda_0\theta_{i-1}t^{\alpha+1}}{\alpha+1} = U^{\alpha+1}$

which implies

$$t = \left(\frac{\alpha+1}{\lambda_0\theta_{i-1}}\right)^{\frac{1}{\alpha+1}} U \quad ; \quad dt = \left(\frac{\alpha+1}{\lambda_0\theta_{i-1}}\right)^{\frac{1}{\alpha+1}} dU$$

Also let

$$W_i = \left(\frac{\lambda_0\theta_{i-1}}{\alpha+1}\right)^{\frac{1}{\alpha+1}} \tag{2.120}$$

and finally

$$V_i = (\theta_{i-1}W_i)^{-1} \int_0^{W_iT_i} \exp(-U^{\alpha+1})U^{\alpha+1}dU \tag{2.121}$$

W_i is also used to evaluate y_i

$$y_i = \int_0^{T_i} \exp\left(-\frac{\lambda_0\theta_{i-1}t^{\alpha+1}}{\alpha+1}\right)dt \tag{2.122}$$

which can be shown as

$$y_i = W_i^{-1} \int_0^{W_iT_i} \exp(-U^{\alpha+1})dU \tag{2.123}$$

From (2.117) and (2.120) we have

$$W_i^{\alpha+1} - W_{i-1}^{\alpha+1} = \frac{\lambda_0}{\alpha+1} (\theta_{i-1} - \theta_{i-2}) = \frac{\lambda_0\epsilon}{\alpha+1} y_{i-1}$$

or

$$W_i^{\alpha+1} = W_{i-1}^{\alpha+1} + \frac{\lambda_0\epsilon}{\alpha+1} y_{i-1} \qquad\qquad i = 2,3,\ldots,N$$

$$W_1^{\alpha+1} = \frac{\lambda_0}{\alpha+1} \tag{2.124}$$

The following computation procedure summarizes the results obtained in this section to evaluate the gradients at $\bar{T}^*$ when the underlying failure rate has the form

$$r_i(t|\theta_{i-1}) = \theta_{i-1}r(t)$$

and the age factor is approximated by $\theta_{i-1} = 1 + \epsilon \sum_{j=1}^{i-1} y_j$

STEP 0: Set $i = 1$, $\theta_0 = 1$

STEP 1: Calculate $y_i = \int_0^{T_i} \bar{F}(t|\theta_{i-1})dt$

STEP 2: Compute $\theta_i = 1 + \sum_{j=1}^{i} y_i = \theta_{i-1} + y_i$

STEP 4: If $i < N$, set $i = i+1$ and go to Step 1; else go to Step 5

STEP 5: Calculate matrix d_{ij}; $i,j = 1,2,\ldots,N$ using eq. (2.112)

STEP 6: Calculate $R_i(N,\bar{T})$; $i = 1,2,\ldots,N$, from equations (2.107) and (2.11) using the results obtained in Step 5

STEP 7: Compute $L_i(N,\bar{T})$; $i = 1,2,\ldots,N$ from equation (2.114)

STEP 8: Calculate $R(N,\bar{T})$ using equations (2.92), (2.110) and the results obtained in Step 6

STEP 9: Calculate $L(N,\bar{T}) = \sum_{i=1}^{N} y_i$

STEP 10: Evaluate $C_i(N,\bar{T})$ using equation (2.115) and the results obtained in Steps 6 through 9

Note that if the underlying lifetime distribution is Weibull, then the following changes are made to reduce computation time.

1. W_i is calculated from either (2.120) or (2.124) before Step 2 is evaluated, and it is used to compute V_i from equation (2.121) and y_i from equation (2.123).

2. $R_i(N,\bar{T})$ is evaluated by using equation (2.118).

Generally any optimum search technique, when N=1, requires an initial interval of uncertainty (X_L, X_R). We may define the lower limit by a time before which there would be a small probability that the system could have failed, i.e.

$$Pr\{\tau_1 \leq a\} \leq \delta \tag{2.125}$$

For the Weibull distribution, we have

$$e^{-\dfrac{\lambda_0 a^{\alpha+1}}{\alpha+1}} \geq 1-\delta$$

or finally

$$a \leq \left[\frac{-(\alpha+1)\ln(1-\delta)}{\lambda_0} \right]^{\frac{1}{\alpha+1}} = X_L \tag{2.126}$$

With a similar argument, we find that

$$X_R \leq \left[\frac{-(\alpha+1)\ln\delta}{\lambda_0} \right]^{\frac{1}{\alpha+1}} = X_L \tag{2.127}$$

For $\delta = 0.001$, equations (2.126) and (2.127) become

$$X_L = \left[\frac{0.001(\alpha+1)}{\lambda_0} \right]^{\frac{1}{\alpha+1}} \tag{2.128}$$

and

$$X_R = \left[\frac{6.9(\alpha+1)}{\lambda_0} \right]^{\frac{1}{\alpha+1}} \tag{2.129}$$

If the optimal solution converges to X_R, then from Figure 2.2(a) we can suspect that optimal $T_1(1)$ is infinity.

Numerical Example 2.4

We now consider the numerical example 2.3, but we employ the approximation method to solve it. The results are shown in Figure 2.5.
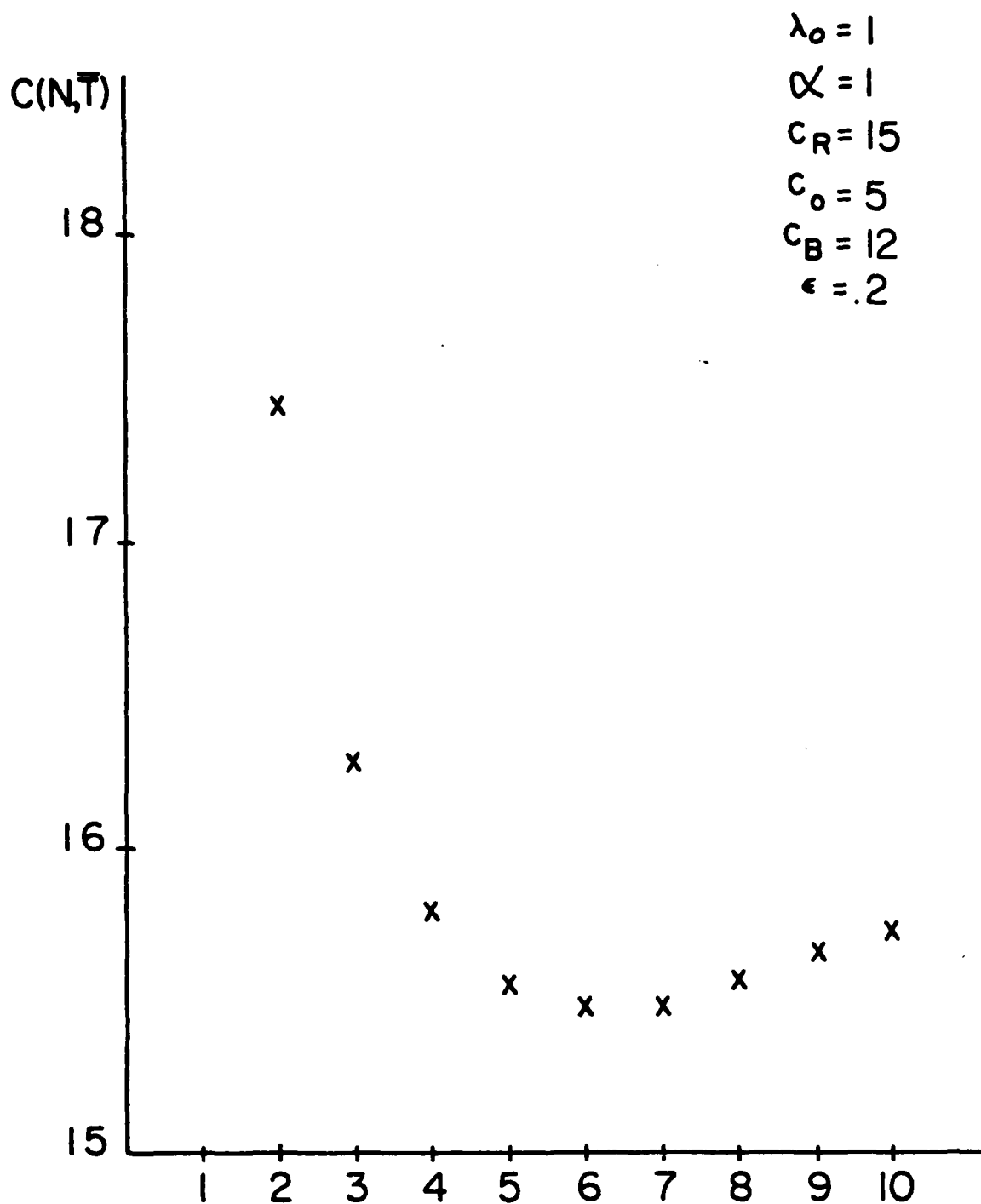
Figure 2.5 – optimal policy I using simulation

The optimal policy I in this case is: $N^* = 6,7$ with $C(N,\bar{T}^*) = 15.49$ and $\bar{T}^* = (.96\ .90\ .85\ .81\ .78\ .74)$ or $\bar{T}^* = (.94\ .89\ .83\ .78\ .75\ .72\ .69)$. We have used the conjugate gradient technique and the Davidson, Fletcher, and Powell method. Obviously, the conjugate gradient technique needs less computation time. The result of lemma 2.1 is also applied to this case, and computations could have been reduced by obtaining an estimate for N^* (say N'^*). This is achieved by finding the minimum of $C(N,\infty)$.

	N	Simulation	Approx.
		$C(N,\infty)$	
	1	21.45	21.45
	2	18.63	18.53
	3	17.63	17.91
$N'^* \rightarrow$	4	17.54	17.84
	5	17.62	17.96
	6	17.99	18.17
	7	18.23	18.43
	8	18.49	18.69
	9	18.88	18.97

Table 2.2.　Comparison between simulation and approximation when $\bar{T} \rightarrow \infty$

Table 2.2 shows that $N'^* = 4$, which could have served as an initial value for N and iterations 1, 2, and 3, could have been avoided. The expected value of each period in this case is

$$\mu_i = \left(\frac{1}{\alpha+1}\right)^{\frac{\alpha}{\alpha+1}} \left(\theta_{i-1}\lambda_0\right)^{-\frac{1}{\alpha+1}} \Gamma(\tfrac{1}{\alpha+1}) \qquad (2.130)$$

where $\theta_{i-1} = 1 + \epsilon \sum\limits_{j=1}^{i-1} \mu_j$.

Table 2.2 compares the limiting behavior of $C(N,\infty)$, as obtained by simulation, to the values that result from the "mean value" approximation (numerical examples 2.3 and 2.4).

2.10 AVAILABILITY

Throughout our discussion about policy I, we assumed that the durations of times for maintenance and renewal are negligible. In a sense, this does not seem to be justifiable. But by interpreting the costs C_R, C_0 and C_B as the expected replacement, maintenance, and breakdown times, we may formulate the type I policy in terms of the steady state availability (limiting efficiency) $A(N, \overline{T})$.

$$A(N, \overline{T}) = \frac{\sum_{i=1}^{N} \int_{0}^{T_i} \overline{F}_i(t|\theta_{i-1})dt}{C_R + (N-1)C_0 + C_B \sum_{i=1}^{N} F_i(T_i|\theta_{i-1}) + \sum_{i=1}^{N} \int_{0}^{T_i} \overline{F}_i(t|\theta_{i-1})dt}$$

$$= [1 + C(N, \overline{T})]^{-1}$$

Therefore, maximizing $A(N, \overline{T})$ is equivalent to minimizing $C(N, \overline{T})$.

CHAPTER 3

POLICY II

SCHEDULING FIXED TIMES
FOR MAJOR REPAIRS WHEN
INTERMEDIATE MINOR REPAIRS
ARE ALLOWED

3.1 INTRODUCTION

This chapter is concerned with policy II and its optimalities when
the major repairs are scheduled and minor repairs are carried out when
failures occur. Minimizing the long-term average cost per unit time is
used as a criterion to optimize this strategy. A general definition of
the type II policy is found in Section 3.2. Two different failure rates
(A and B types) are defined in Section 3.3. The problem is formulated
in terms of nonlinear and dynamic programming in Sections 3.5 and 3.6
respectively. Necessary and sufficient conditions of optimality for this
problem are derived in Section 3.7. Section 3.8 contains a review of
the state-dependent model. Sections 9-11 are devoted to various age-
dependent models and their optimalities. A steady state availability
formulation is found in Section 3.12.

3.2 POLICY II

Unlike policy I, which is suitable for simple equipment as des-
cribed in Chapter 2, policy II is applied to expensive, complex, and

multi-item systems (e.g., computers, airplanes, bulbs in a factory, etc.). In maintaining a large and complex system over an infinite time span, the failure of a single component unit does not necessarily call for replacing the entire system. Instead, the system can be restored to operation by replacing that single unit. Generally, any maintenance action of this type is called "minimal repair," after which the system failure rate evolves as it would have if no failure had taken place. A system having an increasing failure rate, after a certain amount of operation time, is subject to renewal and/or overhaul (major repair). Our objective is to determine the optimum maintenance time intervals (time between two successive major repairs) and renewal cycle or simply a cycle which is defined to be the time between two successive replacements. For a complex system and when the first major repair is a replacement, a strategy known as policy II has received the most attention in the literature [Barlow and Proschan (1965)].

The following is a generalization of the type II policy:

Policy II. The system is repaired for the ith time at age $t_i = \sum_{j=1}^{i} T_j$, provided $i < N$; it is replaced when $i = N$. In the case of failure between maintenance actions, a minimal repair is carried out. This process will be repeated indefinitely.

Our assumptions made in Section 2.1 regarding zero time duration for replacements, major and minor (minimal) repairs, lack of queuing problem for maintenance, system's state being observable, and having increasing failure rates are also assumed for policy II.

This strategy considers two kinds of repairs:

1. In the case of failure, the system can be restored to operation by minimal repairs, after which the failure rate evolves as it would have if no failure had taken place.

2. Planned major repairs in $T_1, T_2, \ldots, T_{N-1}$ units of time from the most recent corrective operations (major repairs) and then replacement at age t_N, where t_i is defined to be

$$t_i = \sum_{j=1}^{i} T_j \tag{3.1}$$

The maintenance costs in this model are defined as C_R for each replacement, which is performed at age t_N, C_M for each major repair at $t_1, t_2, \ldots, t_{N-1}$ and C_0 for each minimal repair in time interval (t_{i-1}, t_i); $i = 1, 2, \ldots, N$. These costs are assumed to be constant despite the fact that they might well be functions of age, number of repairs (major and/or minor), and many other factors.

3.3 MODELS A AND B

After each major repair that is carried out to reduce the hazard rate (and hence the number of minimal repairs), the system does not completely recover, and breakdowns are more likely to occur than if a replacement had been performed. Mathematically, this can be described by a failure rate function after the $(i-1)^{st}$ major repair, $r_i(t)$, satisfying

$$r_i(t_{i-1} + \Delta t) \geq r_{i-1}(t_{i-2} + \Delta t) \geq \ldots \geq r_1(t_0 + \Delta t) \tag{3.2}$$

The function r_i might depend explicitly on $i-1$, $t_{i-1}, t_{i-2}, \ldots, t_1$. Figure 3.1 is a graphical representation of relation (3.2). In this case, major repairs will reduce the failure rates such that
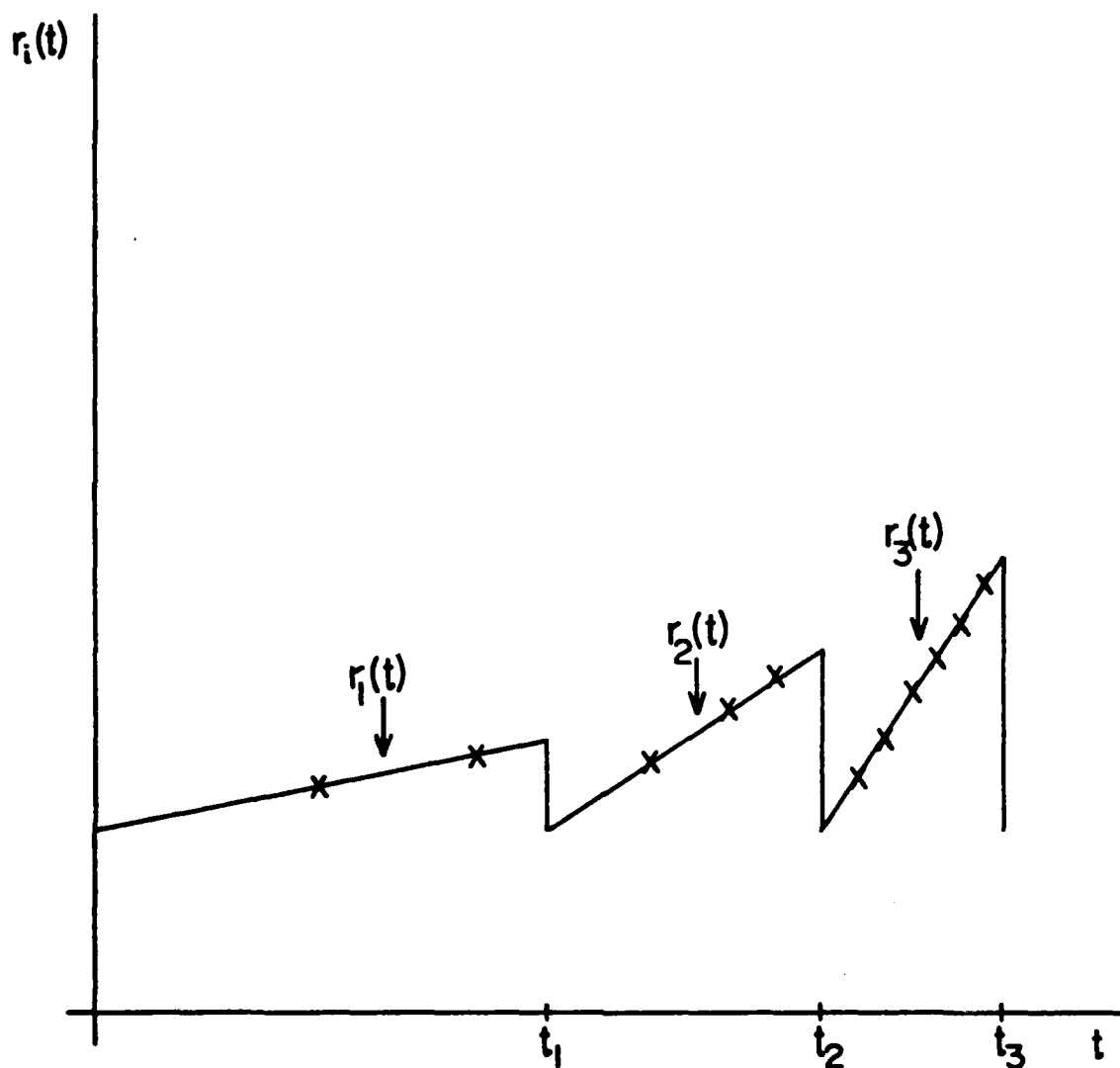
Figure 3.1 - Linearly increasing hazard rates
for Policy II (Model A)

x indicates a failure

$$r_i(t_{i-1}) = r_{i-1}(t_{i-2}) = \ldots = r_1(t_0) \qquad (3.3)$$

We refer to this model as "Model A." Another illustration is shown in Figure 2.1. When relation (3.3) is ignored, one can define "Model B" for which a typical example is represented in Figure (3.2).

3.4 PROBLEM FORMULATION

Let $\theta_i = g_i(T_1, T_2, \ldots, T_{i-1})$ be a positive real function to describe the past history of the system immediately after the $(i-1)^{st}$ preventive maintenance (major repair) action. This definition will permit us to formulate the type II policy when the failure rate depends on the age of the system and/or number of repairs.

The failure characteristic of the system can be expressed by the conditional failure distribution function $F_i(t|\theta_{i-1})$, which is related to the conditional hazard rate $r_i(t|\theta_{i-1})$ and $F_i(t|\theta_{i-1})$ is assumed to be strictly increasing in $T_1, T_2, \ldots, T_{i-1}$, t, and possibly (i-1). We assume that $F_i(t|\theta_{i-1})$ and $r_i(t|\theta_{i-1})$ are both continuous and differentiable.

The expected total cost per renewal cycle, denoted by $R(N, \overline{T})$, is computed by adding the replacement cost, (N-1) major repair costs, and the expected minimal repair cost caused by failures. Based on our notations defined earlier, we have

$$R(N, \overline{T}) = C_R + (N-1)C_0 + C_M \sum_{i=1}^{N} E[N(t_{i-1}, t_i)] \qquad (3.4)$$

where $N(t_{i-1}, t_i)$ is defined to be the number of failures (and hence the number of minimal repairs) occurring in (t_{i-1}, t_i), and

$$\overline{T} = (T_1, T_2, \ldots, T_N) \qquad (3.5)$$
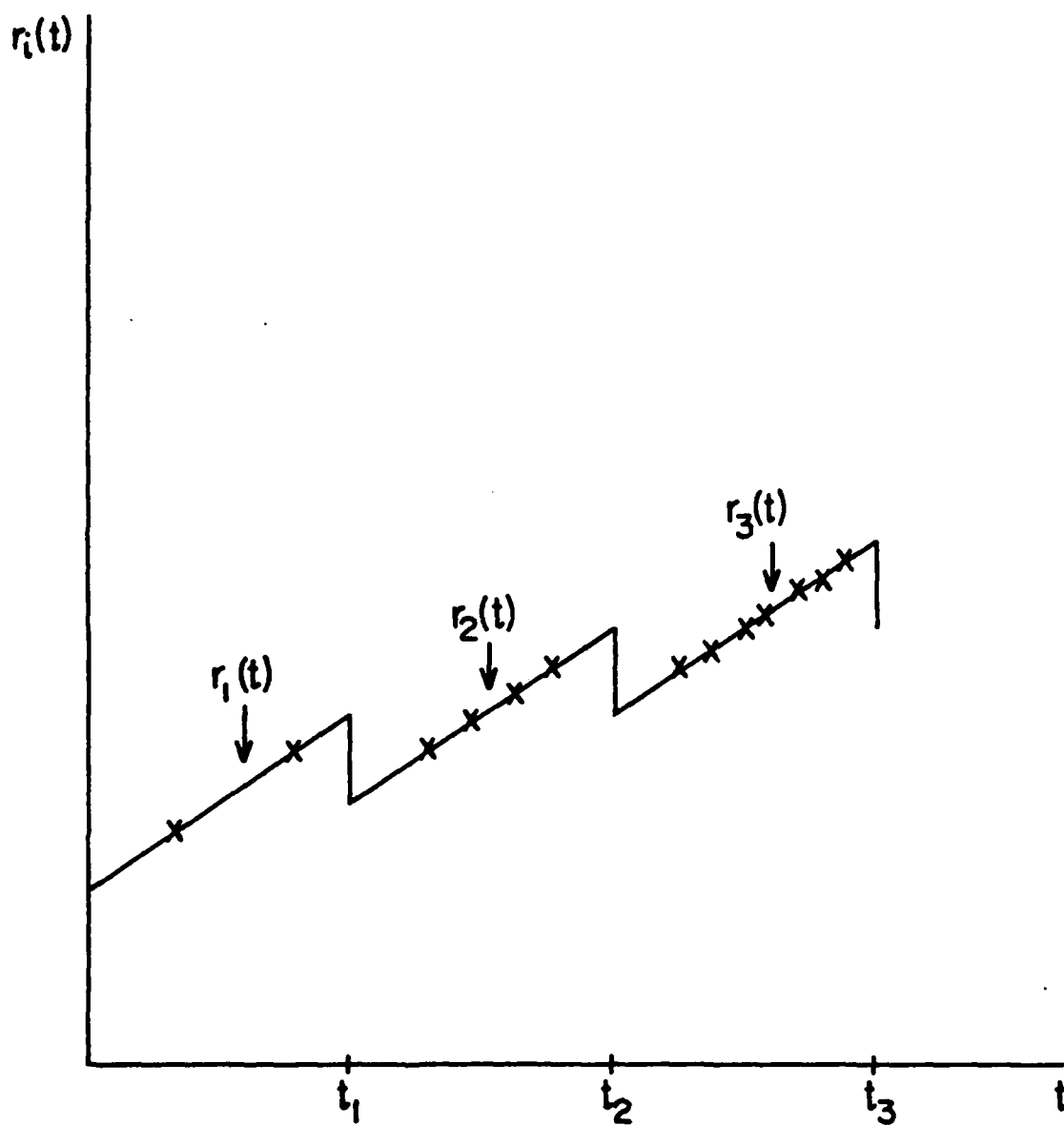
Figure 3.2 – Linearly increasing hazard
rates for Policy II (Model B)

x indicates a failure

By definition, the probability of a failure occurring in (t,t+dt) is $r_i(t|\theta_{i-1})$ + 0(t). Barlow and Hunter (1961) and Barlow and Proschan (1965) have shown that the expected number of failures in (t_{i-1}, t_i) can be expressed as

$$E[N(t_{i-1}, t_i)] = \int_{t_{i-1}}^{t_i} r_i(t - t_{i-1}|\theta_{i-1})dt$$

$$= \int_0^{T_i} t_i(x|\theta_{i-1})dx$$

$$\overset{\Delta}{=} Z_i(T_i|\theta_{i-1}) \tag{3.6}$$

This, of course, follows from the fact that the minimal repairs during time interval (t_{i-1}, t_i) do not disturb the system's failure rate. In other words, after a minimal repair, we assume that the failure has never taken place.

Thus, by using equation (3.6), we can rewrite the identity (3.4) as

$$R(N, \overline{T}) = C_R + (N-1)C_0 + C_M \sum_{i=1}^{N} \int_0^{T_i} r_i(t|\theta_{i-1})dt \tag{3.7}$$

The expected length of a cycle is the total age of the unit

$$L(N, \overline{T}) = t_N = \sum_{i=1}^{N} T_i \tag{3.8}$$

The expected long-term average cost for this model is just the expected cost during a cycle divided by the expected length of a cycle. Therefore, the expected cost per unit time is obtained by

$$C(N, \overline{T}) = \frac{R(N, \overline{T})}{L(N, \overline{T})} = \frac{C_R + (N-1)C_0 + C_M \sum_{i=1}^{N} \int_0^{T_i} r_i(t|\theta_{i-1})dt}{\sum_{i=1}^{N} T_i} \tag{3.9}$$

Note that the hazard rate is affected by age and/or repair only after each major repair, and it is not disturbed by minor repairs which take place before the next major repair.

We now turn our attention to the optimization of preventive maintenance policy II.

3.5 OPTIMAL POLICY II

In order to find an optimal policy II, we seek a set of positive time intervals $(T_i; i = 1,2,\ldots,N)$, which minimizes the expected long-term average cost per unit time given by (3.9). Mathematically, this is expressed by the following nonlinear programming problem (NPII).

Find N^* and $\overline{T}^* = (T_1(N),T_2(N),\ldots,T_N(N))$

To minimize:

$$C(N,\overline{T}) = \frac{R(N,\overline{T})}{L(N,\overline{T})} \tag{3.9}$$

Subject to:

$$T_i \geq 0 \qquad\qquad i = 1,2,\ldots,N \tag{3.10}$$

$$N \geq 1 \tag{3.11}$$

$$N \quad \text{Integer} \tag{3.12}$$

In a way similar to (NPI) given in Chapter 2, we have three special cases to consider.

Case I. No major repairs (N=1)

When major repairs do not result in any improvement or the only possible repair is the minimal repair, then $N^* = 1$. Barlow and Hunter (1960) and Barlow and Proschan have studied this model. They intro-

duced this notion as "periodic replacement with minimal repair at failure." For $N = 1$ equation (3.9) yields

$$C(1,\bar{T}) = \frac{C_R + C_M \int_0^{T_1} r_1(t)dt}{T_1} \tag{3.13}$$

Setting its first derivative to zero, we get the first necessary condition for a minimum

$$C_M T_1 r_1(T_1) - [C_R + C_M \int_0^{T_1} r_1(t)dt] = 0 \tag{3.14}$$

or

$$G_1(T_1) \overset{\Delta}{=} T_1 r_1(T_1) - \int_0^{T_1} r_1(t)dt = \frac{C_R}{C_M} \tag{3.15}$$

Therefore, the optimum time interval between replacements is obtained by $T_1(1)$, which is the solution to equation (3.15). The minimum cost is

$$C^*(1,T_1(1)) = C_M r_1(T_1(1)) \tag{3.16}$$

Barlow and Proschan (1965) have shown that $T_1(1)$ exists and it is unique. This can be seen by differentiating the left-hand side of (3.15) to get

$$\frac{dG_1(T_1)}{dT_1} = T r_1'(T_1) \tag{3.17}$$

which is positive when $r(t)$ is increasing in t. The right-hand side of (3.15) is constant and hence $T_1(1)$ is unique. Figure 3.3 illustrates a typical expected cost function, $C(1,T_1)$.

As an example, suppose the failure rate has the Weibull form

$$r_1(t) = \lambda_0 t^\alpha \qquad t > 0 \ ; \ \alpha > 0 \tag{3.18}$$
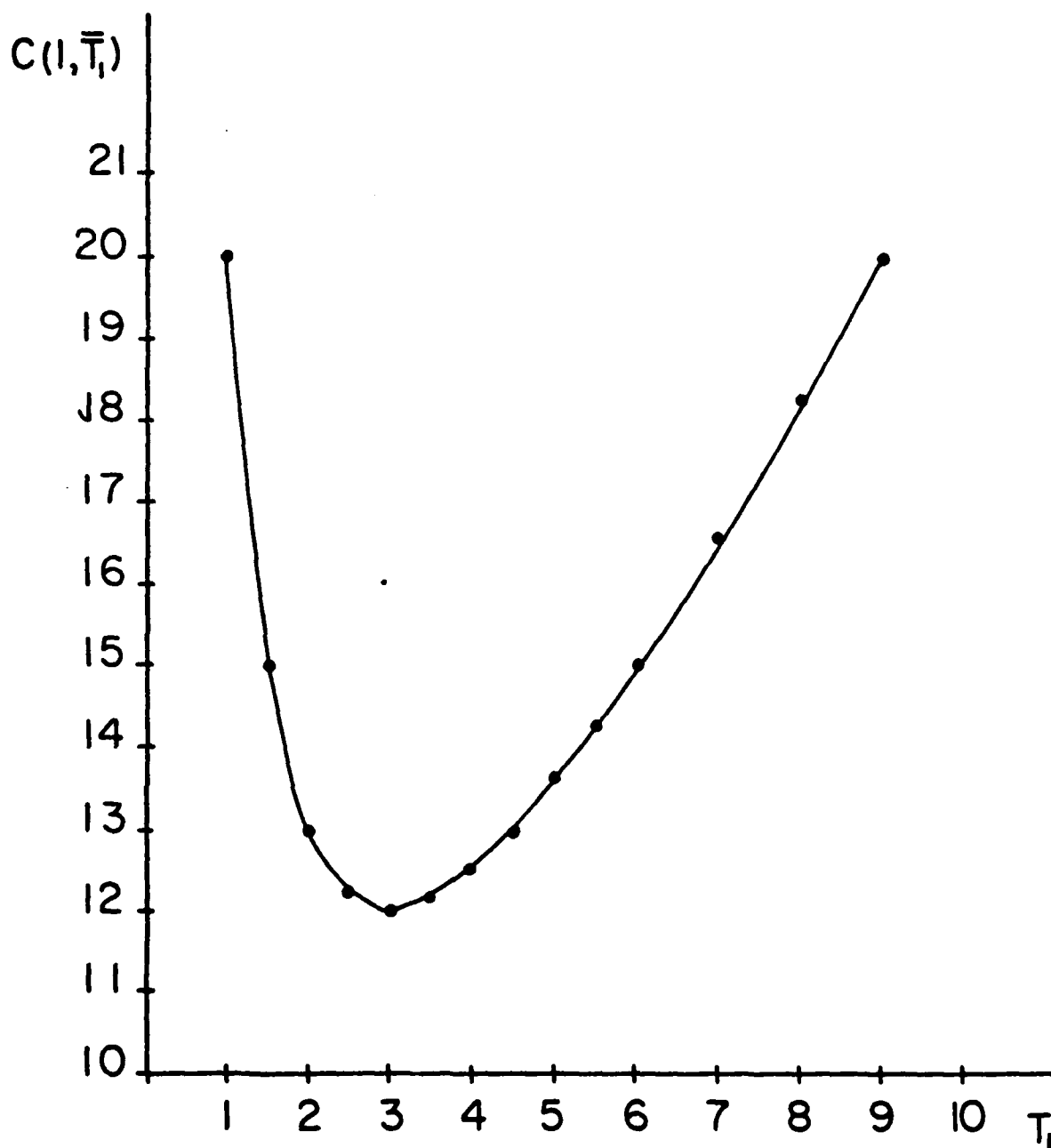
Figure 3.3 – Cost function for
Policy II, N=1

Equation (3.15) can be written as

$$\lambda_0 T_1^{\alpha+1} - \frac{\lambda_0}{\alpha+1} T_1^{\alpha+1} = \frac{C_R}{C_M} \tag{3.19}$$

or

$$\frac{\lambda_0 \alpha}{\alpha+1} T_1^{\alpha+1} = \frac{C_R}{C_M} \tag{3.20}$$

and the optimum time period between replacements is given by

$$T_1(1) = \left[\frac{(\alpha+1)C_R}{\alpha\lambda_0 C_M} \right]^{\frac{1}{\alpha+1}} \tag{3.21}$$

The resulting minimum expected cost is obtained by using identity (2.16)

$$c^*(1,T_1(1)) = C_M \lambda_0 \bigl(T_1(1)\bigr)^\alpha = C_M \lambda_0 \left[\frac{(\alpha+1)C_R}{\alpha\lambda_0 C_M} \right]^{\frac{1}{\alpha+1}} \tag{3.22}$$

Case II. Rapid Aging

This case is similar to Case II described in Chapter 2. It is applied when major repairs have a destructive nature or the system deteriorates very rapidly. As mentioned in Chapter 2, the optimal number of major repairs is $N^* = 1$. Note that, in this case, minimal repairs are performed for continuing operations, and they do not disturb the failure rate.

Case III. Slow Aging

Among the two kinds of repairs, minor repairs do not affect the system's failure rate, but major repairs will result in improvement. This

case assumes that after each major repair, the system becomes as good as new. A mathematical representation of this model is described in terms of hazard rate

$$r_i(T|\theta_{i-1}) = r(t) \qquad (3.23)$$

and the resulting long-term average cost per unit time given by relation (3.9) becomes

$$C(N,\bar{T}) = \frac{C_R + (N-1)C_0 + C_M \sum_{i=1}^{N} \int_0^{T_i} r(t)dt}{\sum_{i=1}^{N} T_i} \qquad (3.24)$$

Setting its first derivatives to zero to get the first condition of optimality

$$C_M r(T_j) \sum_{i=1}^{N} T_i - [C_R + (N-1)C_0 + C_M \sum_{i=1}^{N} \int_0^{T_i} r(t)dt] = 0 \qquad (3.25)$$

$$j = 1,2,\ldots,N$$

or

$$r(T_j) \sum_{i=1}^{N} T_i - \sum_{i=1}^{N} \int_0^{T_i} r(t)dt = \frac{C_R + (N-1)C_0}{C_M} \qquad (3.26)$$

$$j = 1,2,\ldots,N$$

From (3.25), minimum cost is found to be

$$C(N,\bar{T}^*) = C_M r(T_j(N)) \qquad j = 1,2,\ldots,N \qquad (3.27)$$

which yields

$$r(T_j(N)) = r(T_i(N)) \qquad i,j = 1,2,\ldots,N \qquad (3.28)$$

Since r(t) is an increasing function of t, clearly

$$T_j(N) = T_i(N) = T \qquad\qquad i,j = 1,2,\ldots,N \qquad (3.29)$$

Combining (3.29) and (3.26), we get

$$NTr(t) - \int_0^T r(t)dt = \frac{C_R + (N-1)C_0}{C_M} \qquad (3.30)$$

Thus the optimum time interval between two major repairs (or possibly replacements) is obtained by T, which is the solution to

$$G(T) \triangleq Tr(T) - \int_0^T r(t)dt = \frac{C_R + (N-1)C_0}{NC_M} = h(N) \qquad (3.31)$$

Differentiating G(T) with respect to T yields

$$\frac{dG(T)}{dT} = Tr'(T) \geq 0 \qquad (3.32)$$

The right-hand side of (3.31), h(N), is increasing in N if

$$C_R > C_0 \qquad (3.33)$$

In an argument similar to the one given for policy I (Chapter 2, special Case III), we conclude the following:

1. If major repair cost C_0 is more than replacement C_R, then the optimum number of preventive maintenance actions is $N^* = 1$ (the same as Case I).

2. If major repair cost is less than replacement cost, then $N^* = \infty$, and the optimum time interval between major repairs is T, which is the solution to

$$Tr(T) - \int_0^T r(t)dt = \frac{C_0}{C_M} \qquad (3.34)$$

3. The optimum number of planned major repairs is independent of minimal repair costs, C_M. But larger time interval for each scheduled major repair is obtained by smaller C_M.

4. When $C_R = C_0$, then N^* can be any positive integer number.

The above results are somehow obvious. But for state, age, and state-age-dependent policy II, we assume that $C_R > C_0$. This allows us to investigate the existence of nontrivial optimal solutions. Case III is a special case of state-dependent model, developed by Nguyen and Murthy (1981), which we shall briefly discuss in Section 3.7.

Before considering some more general models, a reformulation of the optimal policy II, using the dynamic programming technique, is presented.

3.6 DYNAMIC PROGRAMMING FORMULATION

This problem is analogous to the one discussed in Section 2.4. Equations (2.39), (2.40), (2.41), (2.45), and (2.46) still hold. Identity (2.42) will change to

$$C_K = (1-\eta_K)C_R + \eta_K C_0 + C_M \int_0^{T_K} r(t|\theta_{K-1})dt - \beta T_K \qquad (3.35)$$

where

$$\eta_K = 1 \qquad \text{if} \qquad 1 \le K < N$$
$$= 0 \qquad \text{if} \qquad K = N$$

A typical transition equation can be defined by θ_K when it is proportional to the system's age at the time of the most recent major repair

$$\theta_K = \theta_{K-1} + T_K \qquad K \ne 0$$
$$\theta_0 = 1 \qquad (3.36)$$

where $\in$ is a constant to determine how fast the system deteriorates. This relation is especially suitable when hazard rate has the form

$$r_i(t|\theta_{i-1}) = \theta_{i-1}r(t) \qquad (3.37)$$
$$i = 1,2,\ldots,N$$

which reflects the A-type model defined in Section 3.2. For B-type model θ_0 can take on any nonnegative value.

When failure times follow the Weibull distribution, obtaining optimal policy II by means of dynamic programming requires numerical computations. At each stage, an equation involving parameters like θ_K and β must be solved. Bounds on θ_K and β may reduce inefficiency. A more efficient method is to apply a search technique (e.g. gradient method). For this to be the case, the nature of optimal solution(s) must be known.

3.7 EXISTENCE OF MINIMUM

We have seen that there always exists a minimum when $N = 1$. The following notations are used to investigate the existence and the nature of any stationary point for (NPII) when $N \geq 2$.

Let the first and second partial derivatives of total cost per cycle, $R(N,\bar{T})$, be

$$R_i(N,\bar{T}) = \frac{\partial R(N,\bar{T})}{\partial T_i} \qquad i = 1,2,\ldots,N \qquad (3.38)$$

$$\text{and } R_{ij}(N,\bar{T}) = \frac{\partial^2 R(N,\bar{T})}{\partial T_i \partial T_j} \qquad i,j = 1,2,\ldots,N \qquad (3.39)$$

Also let

$$C_i(N,\bar{T}) = \frac{\partial C(N,\bar{T})}{\partial T_i} \qquad i = 1,2,\ldots,N \qquad (3.40)$$

and $C_{ij}(N,\overline{T}) = \dfrac{\partial^2 C(N,\overline{T})}{\partial T_i \partial T_j}$ $\qquad i,j = 1,2,\ldots,N$ $\qquad\qquad$ (3.41)

be the first and second partial derivatives of $C(N,\overline{T})$.

For $r_K(t|\theta_{K-1})$, since the subscript represents the state of the system (number of major repairs performed), we employ superscripts to distinguish the first and second partial derivatives. Thus

$$r_K^{(i)}(t|\theta_{K-1}) = \frac{\partial r_K(t|\theta_{K-1})}{\partial T_i} \qquad i = 1,2,\ldots,K-1 \qquad (3.42)$$

$$r_K^{(ij)}(t|\theta_{K-1}) = \frac{\partial^2 r_K(t|\theta_{K-1})}{\partial T_i \partial T_j} \qquad i,j = 1,2,\ldots,K-1 \qquad (3.43)$$

Note that r_K is independent of $T_K, T_{K-1}, \ldots, T_N$.

To minimize $C(N,\overline{T})$, we set its derivatives equal to zero

$$C_i(N,\overline{T}) = \frac{R_i(N,\overline{T})}{\displaystyle\sum_{j=1}^{N} T_j} - \frac{R(N,\overline{T})}{\left(\displaystyle\sum_{j=1}^{N} T_j\right)^2} = 0 \qquad (3.44)$$

$$i = 1,2,\ldots,N$$

The first necessary condition for minimum is obtained by

$$R_i(N,\overline{T}) \sum_{j=1}^{N} T_j - R(N,\overline{T}) = 0 \qquad (3.45)$$

or

$$R_i(N,\overline{T}) = \frac{R(N,\overline{T})}{\displaystyle\sum_{j=1}^{N} T_j} = C(N,\overline{T}) \qquad (3.46)$$

$$i = 1,2,\ldots,N$$

which gives

$$R_i(N,\overline{T}) = R_j(N,\overline{T}) \qquad i,j = 1,2,\ldots,N \qquad (3.47)$$

Therefore, the optimal solution is such that the change in total expected minimal cost per cycle will remain the same when both T_i and T_j $(i \neq j)$ are incremented by one unit. From (3.7) $R_i(N, \bar{T})$ is found to be

$$R_i(N, \bar{T}) = C_M \left[r_i(T_i | \theta_{i-1}) + \sum_{j=i+1}^{N} \int_0^{T_j} r_j^{(i)}(t | \theta_{j-1}) dt \right]$$

(3.48)

$$i = 1, 2, \ldots, N$$

Note that $\sum\limits_{j=i+1}^{N} (\ ) = 0$ when $i = N$.

The minimum long-term average cost per unit time can be expressed in terms of the failure rate at the end of the last stage by using relations (3.48) and (3.46) when $i = N$.

$$C(N, \bar{T}^*) = C_M r_N(T_N^* | \theta_{N-1})$$

(3.49)

One can substitute (3.48) and (3.7) into equation (3.45) to get

$$C_M \left[r_i(T_i | \theta_{i-1}) + \sum_{j=i+1}^{N} \int_0^{T_j} r_j^{(i)}(t | \theta_{j-1}) dt \right] \sum_{j=1}^{N} T_j$$

$$- C_R - (N-1)C_0 - C_M \sum_{j=1}^{N} \int_0^{T_j} r_j(t | \theta_{j-1}) dt = 0$$

(3.50)

which can be simplified to

$$\left[r_i(T_i | \theta_{i-1}) + \sum_{j=i+1}^{N} \int_0^{T_j} r_j^{(i)}(t | \theta_{j-1}) dt \right] \sum_{j=1}^{N} T_j$$

$$- \sum_{j=1}^{N} \int_0^{T_j} r_j(t | \theta_{j-1}) dt = \frac{C_R + (N-1)C_0}{C_M}$$

(3.51)

$$i = 1, 2, \ldots, N$$

Setting i = N yields

$$r_N(T_N|\theta_{N-1}) \sum_{j=1}^{N} T_j - \sum_{j=1}^{N} \int_0^{T_j} r_j(t|\theta_{j-1})dt = \frac{C_R + (N-1)C_0}{C_M} \qquad (3.52)$$

Combining (3.47) and (3.48) leads to

$$r_i(T_i|\theta_{i-1}) - r_j(T_j|\theta_{j-1}) =$$

$$\sum_{K=j+1}^{N} \int_0^{T_K} r_K^{(j)}(t|\theta_{K-1})dt - \sum_{K=i+1}^{N} \int_0^{T_K} r_K^{(i)}(t|\theta_{K-1})dt \qquad (3.53)$$

$$i,j = 1,2,\ldots,N$$

(3.53) along with (3.52) constitutes a system of N simultaneous, non-linear algebraic equations with N unknowns which can be solved to obtain the optimal solution(s). This is also achieved by solving (3.51) but with more calculations.

To evaluate the second partial derivatives, we define $K = \text{Min}(i,j)$ and $m = \text{Max}(i,j)$ for each pair of (i,j) to obtain

$$R_{ij}(N,\overline{T}) = C_M \left[r_m^{(K)}(T_m|\theta_{m-1}) + \sum_{n=m+1}^{N} \int_0^{T_n} r^{(ij)}(t|\theta_{n-1})dt \right] \qquad (3.54)$$

$$k,j = 1,2,\ldots,N$$

again $\sum_{n=m+1}^{N} (\ ) = 0$ when $m = N$. $C_{ij}(N,\overline{T})$ is found by using eq. (3.44).

$$C_{ij}(N,\overline{T}) = \frac{R_{ij}(N,\overline{T}) \sum_{m=1}^{N} T_m + R_i(N,\overline{T}) - R_j(N,\overline{T})}{(\sum_{m=1}^{N} T_m)^2}$$

$$- \frac{2[R_i(N,\overline{T}) \sum_{m=1}^{N} T_m - R(N,\overline{T})]}{(\sum_{m=1}^{N} T_m)^3} \qquad (3.55)$$

$$i,j = 1,2,\ldots,N$$

Sufficient conditions for minimum are considerably complex if we deal with Hessian matrix

$$H = [C_{ij}(N,\overline{T})]$$

where $C_{ij}(N,\overline{T})$ is given by (3.55). But these conditions can be simplified if we evaluate H at a stationary point. Since (3.45) and (3.47) will hold for the optimal solution, we can substitute them into the equation (3.55) to get

$$C_{ij}(N,\overline{T}^{*}) = \frac{R_{ij}(N,\overline{T}^{*})}{\sum\limits_{m=1}^{N} T_m(N)} = \frac{R_{ij}^{*}}{L^{*}} \qquad (3.56)$$

The following analysis is used to discuss the sufficient conditions, for which the N-dimensional function $C(N,\overline{T})$ is optimal [Kiepert (1910)]. Let the set of determinants $|D_i|$; $i = 1,2,\ldots,N$ be

$$|D_i| = \begin{vmatrix} \dfrac{R_{11}^{*}}{L^{*}} & \dfrac{R_{12}^{*}}{L^{*}} & \cdots & \dfrac{R_{1i}^{*}}{L^{*}} \\ \dfrac{R_{21}^{*}}{L^{*}} & \dfrac{R_{22}^{*}}{L^{*}} & \cdots & \dfrac{R_{2i}^{*}}{L^{*}} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \dfrac{R_{i1}^{*}}{L^{*}} & \dfrac{R_{i2}^{*}}{L^{*}} & \cdots & \dfrac{R_{ii}^{*}}{L^{*}} \end{vmatrix} \qquad (3.57)$$

Then we have the following

1. $|D_i| < 0$ for $i = 1,3,\ldots$ and $|D_i| > 0$ for $i = 2,4,\ldots$ indicates the presence of a relative maximum at $\overline{T}^{*}$.

2. $|D_i| > 0$ for $i = 1,2,\ldots,N$ represents a minimum at $\overline{T}^{*}$.

3. Failure to satisfy either of these two conditions indicates the presence of saddle point.

Due to the differentiability of $R(N,\overline{T})$ and $C(N,\overline{T})$ with respect to $T_1, T_2, \ldots, T_N$, we have

$$C_{ij}(N,\overline{T}) = C_{ji}(N,\overline{T}) \tag{3.58}$$

and $R_{ij}(N,\overline{T}) = R_{ji}(N,\overline{T})$ \hfill (3.59)

Define

$$|D_i'| = (L^*)^i |D_i| \qquad\qquad i = 1,2,\ldots,N \tag{3.60}$$

Since $L^* > 0$, we can simplify the second order optimality condition by means of $|D_i'|$. In other words, any relation that satisfies $|D_i|$ will also satisfy $|D_i'|$.

$$|D_i| = \begin{vmatrix} R_{11}^* & R_{12}^* & \cdots & R_{1i}^* \\ R_{21}^* & R_{22}^* & \cdots & R_{2i}^* \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ R_{i1}^* & R_{i2}^* & \cdots & R_{ii}^* \end{vmatrix} \tag{3.61}$$

Theorem 3.1.

Under the following conditions:

H1: $r_i(t|\theta_{i-1})$ is strictly increasing in $T_1, T_2, \ldots, T_{i-1}$ and t

H2: $r_m^{(ij)}(t|\theta_{m-1}) \geq 0$ for al i, j and m

$C(N,\overline{T})$ has a positive relative minimum with respect to T_j in the interval $(0,\infty)$, and the solution to the first necessary condition

$$R_i(N,\overline{T}) = C(N,\overline{T}) \tag{3.46}$$

cannot be a relative maximum.

Proof. H2, (3.54), and (3.61) imply that $|D_1^i| > 0$, and hence the relative maximum point cannot exist. $C(N,\overline{T})$ *is a continuous function of* T_i; $i = 1,2,\ldots,N$; *and it approaches infinity as* $\overline{T} \rightarrow 0$ *or* $\rightarrow \infty$. Note that $C(N,\overline{T})$ may possess a saddle point.

The preceding theorem allows us to employ various optimum search techniques to find the minimum. The next theorem is useful for finding proper initial values for $T_1, T_2, \ldots, T_N$.

Theorem 3.2

If (i) $r_i(t|\theta_{i-1})$ is increasing in $T_1, T_2, \ldots, T_N$ and t, and (ii) $r_i^{(m)}(t|\theta_{i-1}) = r_j^{(n)}(t|\theta_{j-1})$ for $m = 1,2,\ldots,i-1$ and $n = 1,2,\ldots,j-1$, then failure rates evaluated at optimal T_i are ordered according to

$$r_i(T_i|\theta_{i-1}) \gtrless r_j(T_j|\theta_{j-1}) \quad ; \quad i > j \tag{3.62}$$

Proof. Without loss of generality, we let $j = i-1$, and from (3.53) and hypothesis (ii), we get

$$r_i(T_i|\theta_{i-1}) - r_{i-1}(T_{i-1}|\theta_{i-2}) = \int_0^{T_i} r_i^{(i-1)}(t|\theta_{i-1})dt \tag{3.63}$$

Condition (i) implies that the right-hand side of (3.63) is positive and hence

$$r_i(T_i|\theta_{i-1}) \gtrless r_{i-1}(T_{i-1}|\theta_{i-2}) \qquad i = 1,2,\ldots,N$$

As an example for which conditions of Theorem 3.2 satisfy, consider the Weibull failure rate

$$r_i(t|\theta_{i-1}) = \lambda_0 \theta_{i-1} t^\alpha \qquad\qquad \alpha \geq 1$$

$$\theta_{i-1} = 1 + \epsilon \sum_{j=1}^{i-1} T_j$$

Note that $r_i^{(m)} = r_j^{(n)} = \epsilon t^\alpha$; $\alpha \geq 1$.

So far, we have studied the general policy II. Now we turn out attention to state- and age-dependent type II policy separately.

3.8 STATE-DEPENDENT MODEL

In this case of policy II, the state represents the number of major repairs performed. The failure rate at each stage is an increasing function of the number of major repairs already carried out and the time from the most recent major repair. Mathematically

$$r_i(t|\theta_{i-1}) = r_i(t) \qquad\qquad t > 0; \ i = 1,2,\ldots,N \qquad (3.64)$$

From (3.9), the long-term average cost per unit time is found to be

$$C(N,\bar{T}) = \frac{C_R + (N-1)C_0 + C_M \sum\limits_{i=1}^{N} \int_0^{T_i} r_i(t)dt}{\sum\limits_{i=1}^{N} T_i} \qquad (3.65)$$

D.G. Nguyen and D.N.P. Murthy (1981) have studied this model for both policy I and policy II. They have shown that these two models have similar behaviors. Their results are summarized as follows:

1. Differentiating (3.65) with respect to t_i and equating it to zero shows that the optimum preventive schedule times satisfy

$$r_i(T_i) = r_1(T_1) \qquad\qquad i = 2,3,\ldots,N \qquad (3.66)$$

and

$$r_1(T_1) \sum_{i=1}^{N} T_i - \sum_{i=1}^{N} \int_0^{T_i} r_i(t)dt = \frac{C_R + (N-1)C_0}{C_M} \qquad (3.67)$$

and the resulting minimum cost is

$$C(N,\overline{T}^*) = C_M r_i(T_i) \qquad (3.68)$$

2. Assumptions (i), (ii), and (iii) made in Section 2.5 will guarantee the existence and uniqueness of an optimal N and optimum planned major repair time intervals $T_i(N)$; $i = 1,2,\ldots,N$. For a fixed N, $T_i(N)$ has been shown to be decreasing in i.

3. When failure times are assumed to be given by Weibull distributions

$$f_i(t) = \lambda_i t^\alpha \exp(- \frac{\lambda_i}{\alpha+1} t^{\alpha+1}) \qquad \alpha \geqq 1 \; ; \; t > 0 \qquad (3.69)$$

Then $T_i(N)$ can be found analytically by letting

$$\eta_i = \left(\frac{\lambda_1}{\lambda_i} \right)^{1/\alpha} \qquad (3.70)$$

which yields

$$T_1(N) = \left[\frac{(N-1)C_o+C_R}{C_M \alpha \lambda_1 \sum\limits_{i=1}^{N} \eta_i(\alpha+1)} \right]^{1/(\alpha+1)}$$

and

$$T_i(N) = \eta_i T_1(N); \qquad\qquad i = 1,2,\ldots,N \qquad (3.71)$$

4. Nguyen and Murthy have also shown that the computation algorithm used for policy I (see Section 2.5) can be applied to policy II, except that here Step (ii) is computed by solving (3.66) and (3.67), and Step (v) is obtained by (3.65). But our heuristic procedure to find approximate N^* for policy I cannot be used for policy II.

3.9 AGE-DEPENDENT MODEL

Unlike policy I in which the age at the time of repair was a random variable, here the age of the system is known and constant when a major repair is performed, and it can be represented by

$$t_i = \sum_{j=1}^{i} T_j \qquad\qquad i = 1,2,\ldots,N \qquad\qquad (3.72)$$

where t_N indicates the replacement time. We consider two types of age-dependent policy II as follows:

1. One corresponds to "Model A" defined in Section 3.2. In this case, the hazard rate at the beginning of each stage is given by

$$r_i(t|\theta_{i-1}) = \theta_{i-1} r(t) \qquad\qquad i = 1,2,\ldots,N \qquad\qquad (3.73)$$
$$t \geqq 0; \ \theta_0 \geqq 1$$

As mentioned earlier, θ_{i-1} is an increasing function of $T_1, T_2, \ldots, T_{i-2}$.

2. In the second model, the failure rates have the form

$$r_i(t|\theta_{i-1}) = r(t) + \theta_{i-1} \qquad\qquad i = 1,2,\ldots,N \qquad\qquad (3.74)$$
$$t \geqq 0 \ ; \ \theta_0 \geqq 0$$

which is a particular case of "Model B."

3.10 MODEL A

The age factor for this case may be defined as

$$\theta_i = 1 + t_i = 1 + \epsilon \sum_{j=1}^{i} T_j \qquad\qquad (3.75)$$

where ϵ is a positive parameter which measures the degree of deterioration as the system gets older. Note that for a new system, $\theta_0 = 1$ and the failure rate (3.73) is described by $r(t)$. From (3.9), we conclude the long-term average cost for unit time

$$C(N,\bar{T}) = \frac{C_R + (N-1)C_0 + C_M \sum\limits_{i=1}^{N} \theta_{i-1} \int_0^{T_i} r(t)dt}{\sum\limits_{i=1}^{N} T_i} \qquad (3.76)$$

The derivative of r_i with respect to T_j in this case is

$$
\begin{aligned}
r_i^{(j)}(t|\theta_{j-1}) &= \in r(t) & j &= 1,2,\ldots,i-1 \\
&= 0 & j &= i,i+1,\ldots,N
\end{aligned}
\qquad (3.77)
$$

which can be substituted into relation (3.51) to get the first necessary condition for minimum

$$[r_i(T_i)\theta_{i-1} + \in \sum_{j=i+1}^{N} \int_0^{T_j} r(t)dt] \sum_{j=1}^{N} T_j - \sum_{j=1}^{N} \theta_{j-1} \int_0^{T_j} r(t)dt$$

$$= \frac{C_R + (N-1)C_0}{C_M} \qquad (3.78)$$

$$i = 1,2,\ldots,N$$

Equation (3.53) becomes

$$r(T_{i+1})\theta_i - r(T_i)\theta_{i-1} = \in \int_0^{T_{i+1}} r(t)dt \qquad (3.79)$$

$$i = 1,2,\ldots,N-1$$

As an example, consider a system whose failures follow the conditional Weibull distributions

$$f_i(t|\theta_{i-1}) = \lambda_0 \theta_{i-1} t^{\alpha} \exp(- \frac{\lambda_0 \theta_{i-1}}{\alpha+1} t^{\alpha+1}) \qquad (3.80)$$

$$i = 1,2,\ldots,N \qquad\qquad \theta_0 = 1; \ t \geq 0; \ \alpha \geq 1$$

with the corresponding conditional hazard rate

$$r_i(t|\theta_{i-1}) = \lambda_o\theta_{i-1}t^{\alpha} \qquad i = 1,2,\ldots,N \qquad (3.81)$$

where θ_i is given by (3.75). In this case, $C(N,\overline{T})$ has the following form:

$$C(N,\overline{T}) = \frac{C_R + (N-1)C_o + \dfrac{C_M\lambda_o}{\alpha+1}\sum\limits_{m=1}^{N}(1+\epsilon\sum\limits_{j=1}^{m-1}T_j)T_m^{\alpha+1}}{\sum\limits_{j=1}^{N}T_j} \qquad (3.82)$$

and equation (3.79) will change to

$$(1 + \epsilon\sum_{j=1}^{i}T_j - \frac{\epsilon}{\alpha+1}T_{i+1})T_{i+1}^{\alpha} = (1 + \epsilon\sum_{j=1}^{i-1}T_j)T_i^{\alpha} \qquad (3.83)$$

$$i = 1,2,\ldots,N-1$$

These are N-1 nonlinear equations. The Nth equation is the result of (3.78) when i=N:

$$(1+\epsilon\sum_{j=1}^{N-1}T_j)T_N^{\alpha}\sum_{j=1}^{N}T_j - \frac{1}{\alpha+1}\sum_{m=1}^{N}(1+\epsilon\sum_{j=1}^{m-1}T_j)T_m^{\alpha+1} = \frac{C_R+(N-1)C_o}{\lambda_oC_M} \qquad (3.84)$$

Thus, one way to obtain the optimum solution is to solve (3.83) and (3.84) simultaneously for $T_1,T_2,\ldots,T_N$. But according to Theorem 3.1 saddle points, but not a relative maximum, are also possible solutions. The following theorem may help us to distinguish minimum point from saddle points.

Theorem 3.3

If the underlying distribution is Weibull with an increasing failure rate and scale parameter

$$\lambda_i = \lambda_o\theta_{i-1} = \lambda_o(1+\epsilon\sum_{j=1}^{i-1}T_j) \qquad (3.85)$$

Then the optimal scheduled repair times, T_i, for policy II are ordered as

$$T_1 \geq T_2 \geq \ldots \geq T_N \tag{3.86}$$

Proof. Consider two arbitrary positive numbers a and b. Let

$$C_a = C(N,\overline{T})|T_i = a, \; T_{i+1} = b$$

and

$$C_b = C(N,\overline{T})|T_i = b, \; T_{i+1} = a$$

where each T_j in C_a and C_b are respectively the same for $j \neq i$, i+1. It is desired to show that for any a > b, we have

$$C_a < C_b \tag{3.87}$$

In relation (3.76), we can see that the total age $\sum_{j=1}^{N} T_j$ and θ_j (j<i and j>i+1) remain unchanged after interchanging the values of T_i and T_{i+1}. Subtracting C_b from C_a yields

$$C_a - C_b = \frac{C_M D}{\sum_{j=1}^{N} T_j}$$

where D is defined to be

$$D = \theta_{i-1}\int_0^a r(t)dt + (\theta_{i-1} + \in a)\int_0^b r(t)dt$$

$$- \theta_{i-1}\int_0^b r(t)dt - (\theta_{i-1} + \in b)\int_0^a r(t)dt$$

$$= \in [a\int_0^b r(t)dt - b\int_0^a r(t)dt]$$

It is sufficient to prove that $D < 0$. Setting $r(t) = \lambda_0 t^\alpha$, we get

$$D = \epsilon [\frac{a}{\alpha+1} b^{\alpha+1} - \frac{b}{\alpha+1} a^{\alpha+1}] = \frac{ab}{\alpha+1} (b^\alpha - a^\alpha)$$

but we assumed that $a > b$, $\alpha \geq 1$ and hence $D < 0$, which implies (3.87).

Another way to find the minimum of $C(N,\bar{T})$ is to apply an optimum search technique. This in turn requires analytical expressions for gradients $C_i(N,\bar{T})$. From (3.48), we have

$$R_i(N,\bar{T}) = C_M[\theta_{i-1} r(t) + \epsilon \sum_{j=i+1}^{N} \int_0^{T_j} r(t)dt] \qquad (3.88)$$

$$i = 1,2,\ldots,N$$

and for Weibull case

$$R_i(N,\bar{T}) = \lambda_0 C_M[(1 + \epsilon \sum_{j=1}^{N} T_j)T_i^\alpha + \frac{\epsilon}{\alpha+1} \sum_{j=i+1}^{N} T_j^{\alpha+1}] \qquad (3.89)$$

$$i = 1,2,\ldots,N$$

$C_i(N,\bar{T})$ can be calculated by

$$C_i(N,\bar{T}) = \frac{R_i(N,\bar{T})L(N,\bar{T}) - R(N,\bar{T})L_i(N,\bar{T})}{[L(N,\bar{T})]^2} \qquad (3.90)$$

$$i = 1,2,\ldots,N$$

To find the second partial derivatives of $R(N,\bar{T})$, let $m = \max(i,j)$ and hence

$$R_{ij}(N,\bar{T}) = \epsilon C_M r(T_m) \quad ; \quad i \neq j$$

$$= C_M \theta_{i-1} r'(T_i) \quad ; \quad i = j \qquad (3.91)$$

Note that conditions H1 and H2 given in Theorem 3.1 are both satisfied.

Numerical Example 3.1

Assume that the failure time of both the new and repaired system have Weibull distributions given by (3.80) with parameters $\alpha=\lambda_o=$ =1. The maintenance costs are: C_R=15.0, C_o=1.0, and C_M=0.3. We applied the gradient method to $C(N,\overline{T})$ (equation (3.82)). The optimum number of major repairs per cycle is N^*=8 with the minimum expected cost per unit time of $C(N^*,\overline{T}^*)$ = 2.88. The optimum major repair times are

$$\overline{T}^* = (7.92, 0.88, 0.83, 0.80, 0.77, 0.74, 0.72, 0.70)$$

Figure 3.4 illustrates the behavior of the $C(N,\overline{T}^*)$ as N is incremented. It is a convex function and its minimum is unique.

In our numerical examples, the convexity of $C(N,T^*)$ with respect to N seems to be a sure possibility. In addition to numerical example 3.1, Figure 3.5 also illustrates this property, but sufficient conditions and proof have yet to be found.

Other observations in all numerical examples we have studied are as follows:

1. The optimum renewal cycle's age, $t_N^* = \sum_{i=1}^{N} T_i(N)$ is an increasing function of N (Figure 3.6).

2. The optimum time periods are such that

$$T_i(N) \geq T_i(N+1) \; ; \; i = 1,2,\ldots,N \tag{3.92}$$

One sufficient condition that certainly must be considered is the fact that the major repair cost, C_o, is less than the replacement cost C_R.
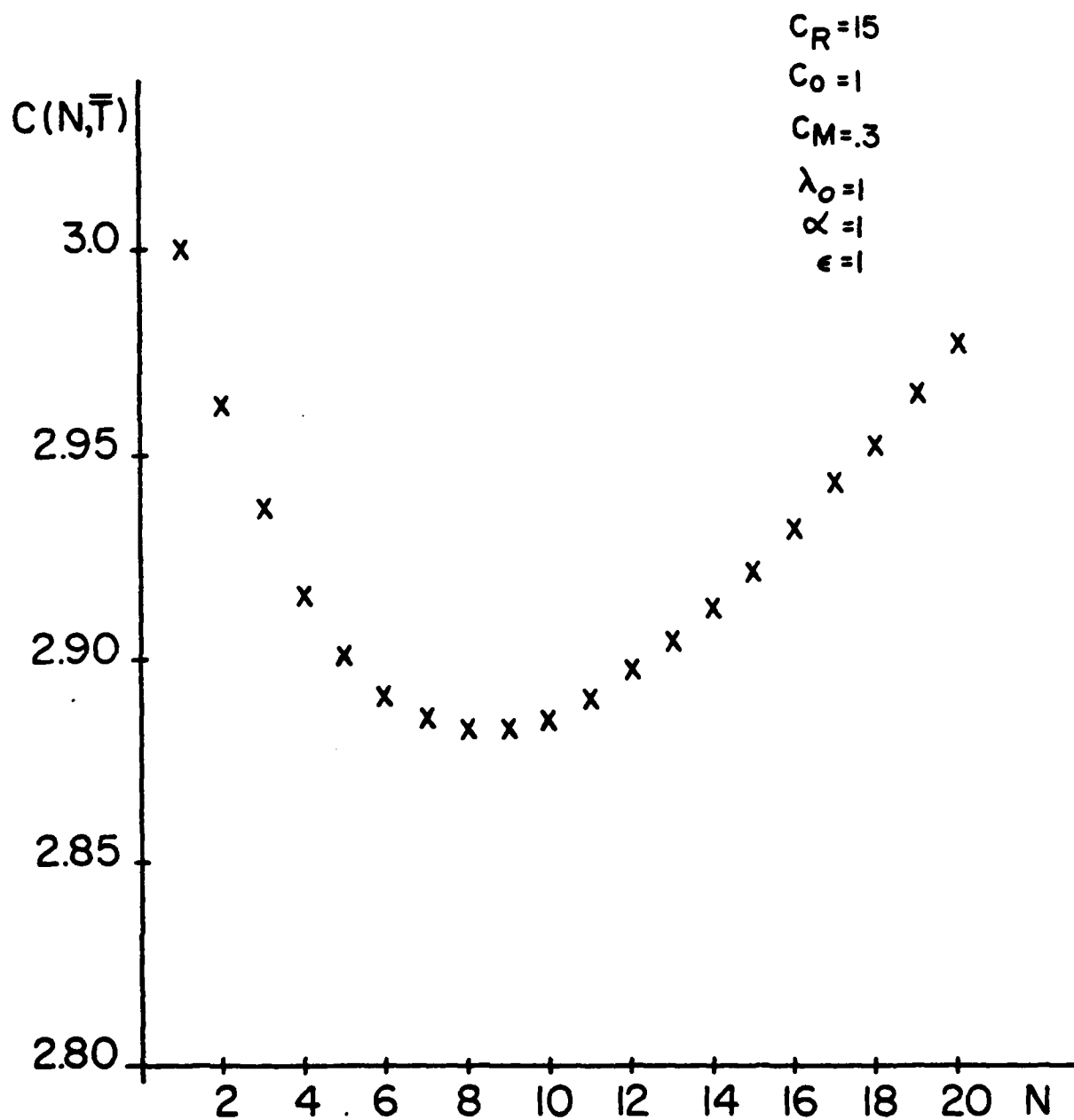
Figure 3.4 - The behavior of the minimum
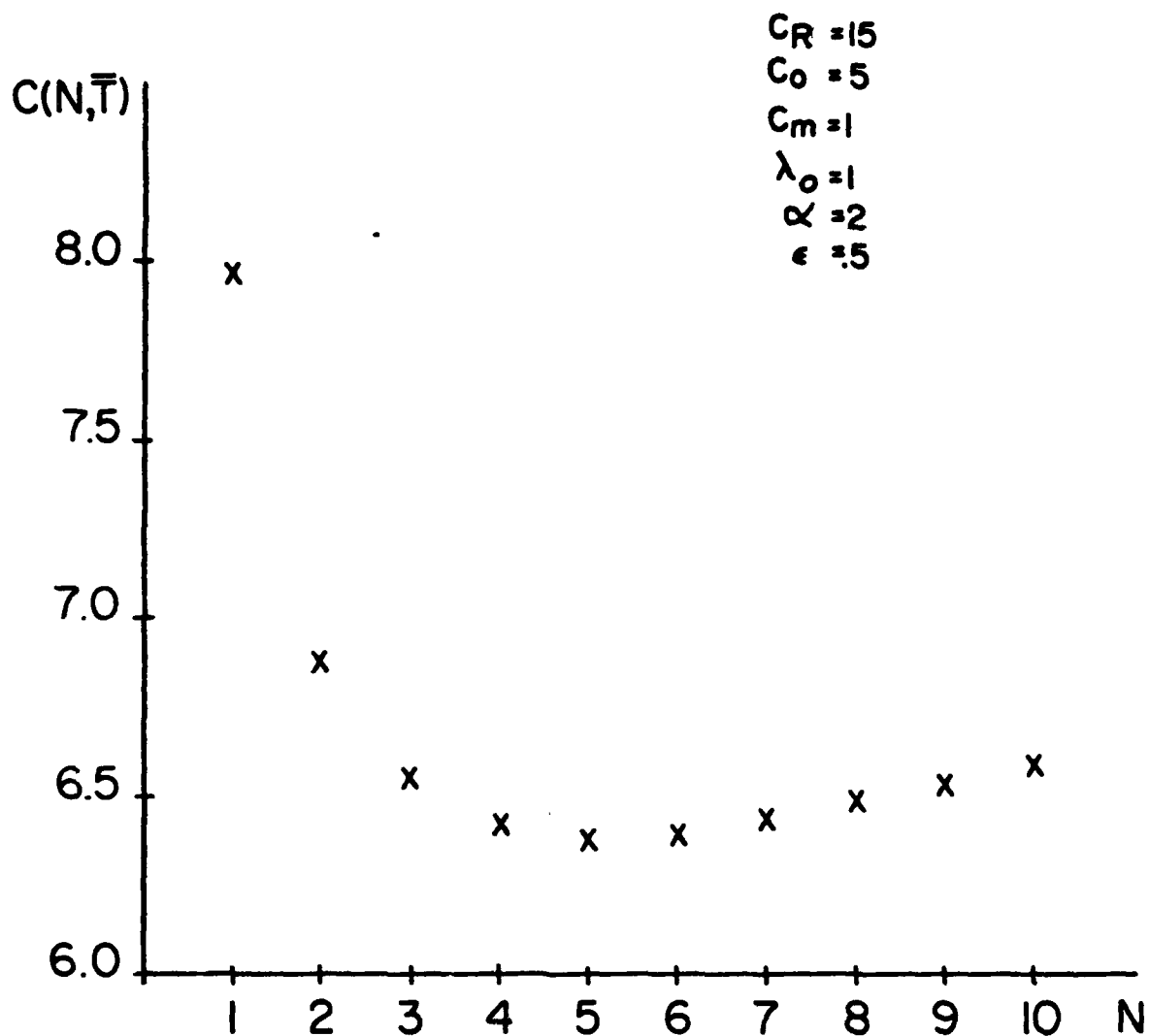cost function an n is
incremented (Policy II - A)

Figure 3.5 - The behavior of the minimum cost
function as N is incremented
(Policy II-A)
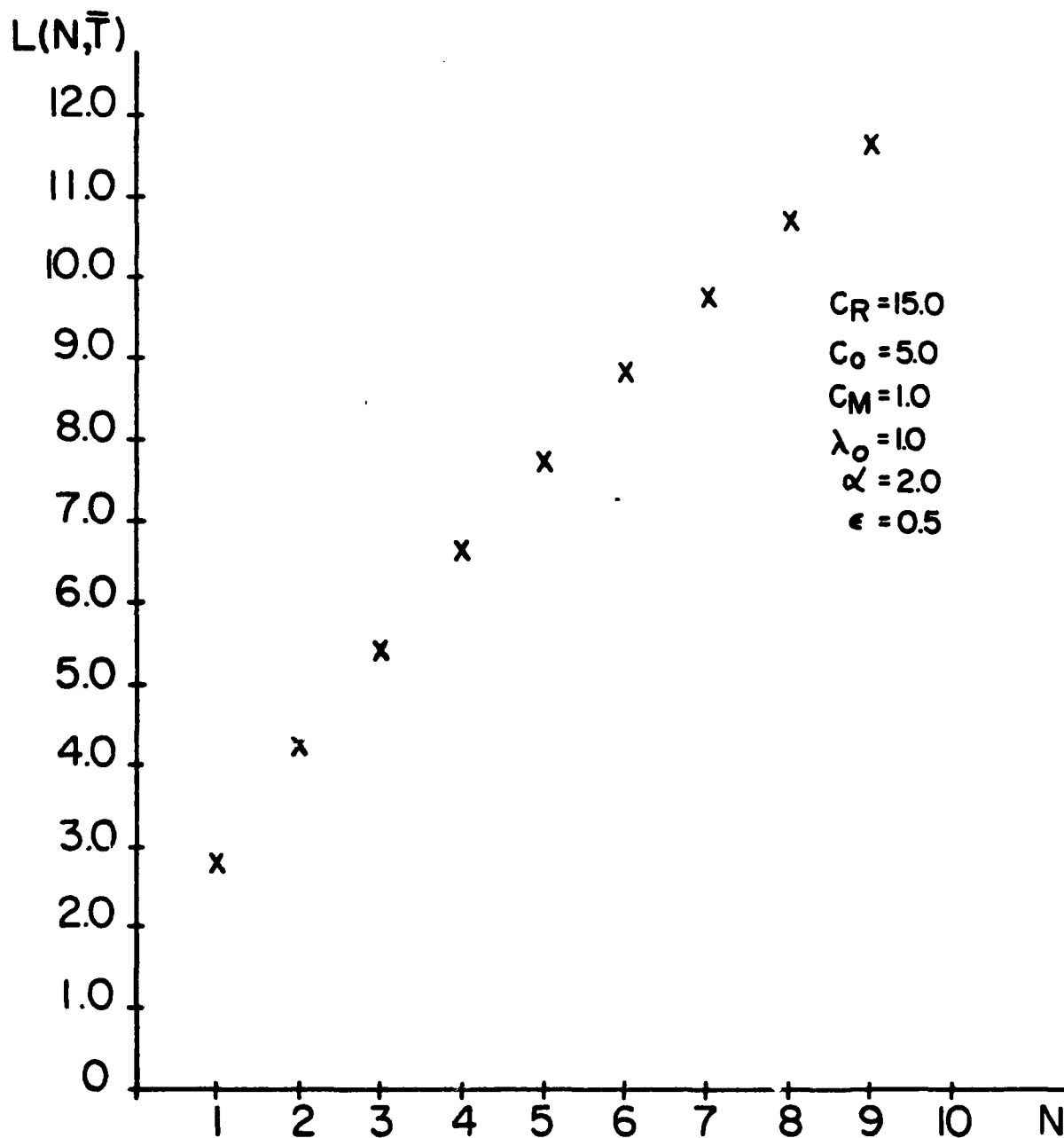
Figure 3.6- The behavior of the optimal cycle
time as N is incremented
(Policy II-A)

In order to reduce computations, initial values for $T_1, T_2, \ldots, T_N$ seem to be helpful.

This is achieved by using Theorem 3.2. For instance, in the Weibull case, the relations (3.62) can be written as

$$r(T_{i+1})\theta_i \geqq r(T_i)\theta_{i-1} \qquad\qquad i = 1, 2, \ldots, N-1 \qquad\qquad (3.93)$$

Setting $r(t) = \lambda_0 t^\alpha$, we get

$$\lambda_0 T_{i+1}^\alpha \theta_i \geqq \lambda_0 T_i^\alpha \theta_{i-1}$$

or $\quad T_{i+1} \geqq \left(\dfrac{\theta_{i-1}}{\theta_i} \right)^{1/\alpha} T_i \qquad\qquad i = 1, 2, \ldots, N-1 \qquad\qquad (3.94)$

where θ_i is given by (3.75).

As an example, consider the numerical example 3.1. The optimal solutions for N=1 and 2 are shown in Table 3.1.

N	$C(N, \overline{T})$	$T_1(N)$	$T_2(N)$
1	3.00	10.00	--
2	2.96	9.48	0.97

Table 3.1

Initially, we used $T_1(2) = 10.0$ and from relation (3.94), a reasonable estimate is $T_2(2) = 0.91$, which seems to be close to 9.48 and 0.97 respectively.

Finally, two reformulations of the age-dependent policy II are presented without any analysis. Despite their additional difficulties caused by the extra number of variables and constraints, some results can be obtained more easily by means of interpretation of the Lagrange
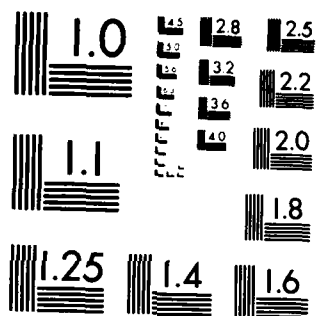
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multipliers. Our objective is to show that they do exist, but no attempt will be made to derive equations satisfying optimal solutions.

The first formulation introduces the age factor at the time of repair as a constraint by letting

$$C_M \theta_{i-1} = A_{i-1} \quad ; \quad i = 2,3,\ldots,N \tag{3.95}$$

Thus, the expected cost per unit time (3.76) can be written as

$$C(N,\overline{T}) = \frac{C_R + (N-1)C_o + \sum\limits_{i=1}^{N} A_{i-1} \int_o^{T_i} r(t)dt}{\sum\limits_{i=1}^{N} T_i} \tag{3.96}$$

This cost function is to be minimized in the following nonlinear programming problem

<u>Find</u>: $N^*, T_1^*, T_2^*, \ldots, T_N^*, A_1^*, A_2^*, \ldots, A_{N-1}^*$

To minimize: $C(N,\overline{T})$ $\tag{3.96}$

Subject to: $C_M \theta_{i-1} = A_{i-1} \quad i = 2,3,\ldots,N$ $\tag{3.97}$

$\overline{T} \geq 0$

$N \geq 1$

$N \quad$ integer

The Lagrange multiplier can be used to analyze this problem. Mathematically, this is equivalent to the one we discussed previously. Lagrange multipliers associated with constraint (3.97) are found to be

$$M_i = \frac{\int_o^{T_i} r(t)dt}{\sum\limits_{j=1}^{N} T_j} \quad ; \qquad i = 1,2,\ldots,N \tag{3.98}$$

The last formulation deals with the following ratio

$$\beta_i = \frac{T_i}{\sum\limits_{j=1}^{N} T_j} = \frac{T_i}{t_N} \tag{3.99}$$

θ_i can be expressed as

$$\theta_i = 1 + \epsilon t_N \sum\limits_{j=1}^{i} \beta_j \tag{3.100}$$

The long-term average cost per unit time is given by

$$C(N,\overline{T}) = \frac{C_R + (N-1)C_o + C_M \sum\limits_{i=1}^{N}(1+\epsilon T_N \sum\limits_{m=1}^{i-1}\beta_m)\int_o^{\beta_i t_N} r(t)dt}{t_N} \tag{3.101}$$

and the resulting nonlinear programming problem is

<u>Find</u>: $N^*, \beta_1^*, \beta_2^*, \ldots, \beta_N^*, t_N$

To minimize: $C(N,\overline{T})$ $\hspace{3cm}$ (3.101)

Subject to: $\sum\limits_{i=1}^{N} \beta_i = 1$ $\hspace{3cm}$ (3.102)

$\hspace{3cm} \overline{T} \geq 0$

$\hspace{3cm} N \geq 1$

$\hspace{3cm} N$ integer $\hspace{2.5cm}$ (3.103)

3.11 MODEL B

The age factor for this model may be defined as

$$\theta_i = \epsilon \sum\limits_{j=1}^{i} T_j \hspace{2cm} i = 1,2,\ldots,N; \quad \geq 0 \tag{3.104}$$

and the failure rate for each time period is

$$r_i(t|\theta_{i-1}) = r(t) + \epsilon \sum_{j=1}^{i-1} T_j \qquad i = 1,2,\ldots,N$$

$$t, \quad \geqq 0 \tag{3.105}$$

where $r_1(t|\theta_{i-1}) = r(t)$.

Figure 3.2 is an illustration of this case when failure rates are linearly increasing. The long-term average cost per unit time is found by combining (3.9) and (3.105).

$$C(N,\bar{T}) = \frac{C_R + (N-1)C_0 + C_M \sum_{i=1}^{N} \int_0^{T_i} r(t)dt + \epsilon C_M \sum_{i=2}^{N} (T_i \sum_{j=1}^{i-1} T_j)}{\sum_{j=1}^{N} T_j} \tag{3.106}$$

Taking its first partial derivatives and setting them to zero, we get:

$$[r(T_i) + \epsilon \sum_{m \neq i}^{N} T_m] \sum_{j=1}^{N} T_j - \sum_{j=1}^{N} \int_0^{T_j} r(t)dt - \epsilon \sum_{j=1}^{N} (T_j \sum_{m=2}^{j-1} T_m)$$

$$= \frac{C_R + (N-1)C_0}{C_M} \qquad i = 1,2,\ldots,N \tag{3.107}$$

From which the minimum expected cost per unit time is obtained by

$$C(N,\bar{T}^*) = C_M[r(T_i) + \epsilon \sum_{m \neq i}^{N} T_m] \qquad i = 1,2,\ldots,N \tag{3.108}$$

which yields

$$r(T_i) + \epsilon \sum_{m \neq i}^{N} T_m = r(T_j) + \epsilon \sum_{m \neq j}^{N} T_m \tag{3.109}$$

or

$$r(T_i) - t(T_j) = \epsilon (T_i - T_j) \tag{3.110}$$

One obvious solution is $T_i = T_m \triangleq T_N^*$. To see if this produces a minimum point, we evaluate $|D_i'|$, which has been defined by (3.61) as

$$|D_i'| = |[R_{Km}]|$$

The second partial derivatives of $R(N,\bar{T})$ are

$$R_{ij} = C_M r' T_i) \qquad ; \qquad j = i \qquad (3.111)$$

$$= C_M \in \qquad ; \qquad j \neq i$$

and consequently

$$|D_i'| = C_m^i \begin{vmatrix} r'(T_1) & \in & \in & \cdots & \in \\ \in & r'(T_2) & \in & \cdots & \in \\ \in & \in & r'(T_3) & \cdots & \in \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \in & \in & \in & & r'(T_i) \end{vmatrix} \qquad (3.112)$$

$$i = 1,2,\ldots,N$$

Setting $T_i = T_j = T_N^*$ for all i and j yields

$$|D_i'| = C_m^i \begin{vmatrix} r'(T_N^*) & \in & \in & \cdots & \in \\ \in & r'(T_N^*) & & \cdots & \in \\ \in & \in & r'(T_N^*) & \cdots & \in \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \in & \in & \in & & r'(T_N^*) \end{vmatrix} \qquad (3.113)$$

$$i = 1,2,\ldots,N$$

which gives

$$|D_i'| = r'(T_N^*)$$

$$|D_2'| = [r'(T_N^*)]^2 - \epsilon^2$$

$$= [r'(T_N^*) - \epsilon][r'(T_N^*) + \epsilon]$$

$$|D_3'| = [r'(T_N^*)]^3 - 3\epsilon^2 r'(T_N^*) + 2\epsilon^3$$

$$= [r'(T_N^*) - \epsilon]^2[f'(T_N^*) + 2\epsilon]$$

and in general

$$|D_i'| = [r'(T_N^*) - \epsilon]^{i-1}[r'(T_N^*) + (i-1)\epsilon] \tag{3.114}$$

In order to have a minimum, it is required that $|D_i'| > 0$ for all i and hence the sufficient condition that $T_i = T_j = T_N^*$ to be a minimum is expressed by

$$r'(T_N^*) > \epsilon \tag{3.115}$$

Setting $T_i = T_N^*$ in relation (3.107), we get an equation that can be solved for T_N^*.

$$[r(T_N^*) + \epsilon(N-1)T_N^*]NT_N^* - N\int_0^{T_N^*} r(t)dt$$

$$- \frac{\epsilon(N-1)N}{2} T_N^{*2} = \frac{C_R + (N-1)C_0}{C_M}$$

or $\quad g_N(T_N^*) \triangleq T_N^* r(T_N^*) - \int_0^{T_N^*} r(t)dt + \frac{\epsilon(N-1)}{2} T_N^{*2}$

$$= \frac{C_R + (N-1)C_0}{NC_M} \triangleq h(N) \tag{3.116}$$

The following theorem is useful for finding the sufficient condition for minimum and an estimate of N^*.

Theorem 3.4

If (i) $r(t)$ is strictly increasing in N and (ii) $C_R > C_0$, then the solution to equation (3.116) is decreasing in N and unique.

Proof. For the right-hand side of (3.116), we have

$$\Delta h(N) = h(N+1) - h(N) = \frac{-C_R + C_0}{N(N+1)C_M} \tag{3.117}$$

From hypothesis (ii), we conclude that

$$\Delta h(N) \leq 0 \tag{3.118}$$

For the right-hand side of (3.116), we can write

$$\Delta g_N(x) = g_{N+1}(x) - g_N(x) = \frac{\epsilon}{2} x^2 \geq 0$$

or $\quad g_{N+1}(x) \geq g_N(x) \tag{3.119}$

which implies that g_N is increasing in N. From (3.118), we have

$$\Delta h(N) = g_{N+1}(T_{N+1}^*) - g_N(T_N^*) < 0$$

or $\quad g_{N+1}(T_{N+1}^*) < g_N(T_N^*) \tag{3.120}$

By using the implication of (3.119), the left-hand side of (3.120) can be decreased to get

$$g_N(T_{N+1}^*) < g_N(T_N^*)$$

and consequently

$$T_{N+1}^* < T_N^* \tag{3.121}$$

T_N^* is unique since $h(N)$ is independent of T_N^* and

$$\frac{dg_N(T_N^*)}{dT_N^*} = T_N^* r'(T_N^*) + \epsilon(N-1)T_N^* \geq 0 \tag{3.122}$$

Hence, T_N^* is unique and decreasing in N.

One application of the preceding theorem is to modify sufficient condition (3.115). Since T_1^* is the largest value amont T_i's, we have

$$r'(T_1^*) > \epsilon \qquad\qquad (3.123)$$

where T_1^* is the solution to equation (3.19).

As an example, consider a new system having Weibull failure time distribution given by

$$f(t) \;=\; \lambda t^\alpha \exp\!\left(-\,\frac{\lambda t^{\alpha+1}}{\alpha+1}\right) \qquad\qquad \alpha \geq 1;\ t > 0;\ \lambda > 0$$

T_1^* is given by (3.21) and hence

$$\alpha\lambda \left[\frac{(\alpha+1)C_R}{\alpha\lambda C_M}\right]^{1/(\alpha+1)} > \epsilon \qquad\qquad (3.124)$$

Therefore, in the Weibull case, if the age deterioration factor, ϵ, is small enough, then the optimal solution is unique and all major repair time intervals are equal.

To obtain the optimal N, we set $T_i = T$ in relation (3.106) to get

$$C(N,\overline{T}) \;=\; \frac{C_R + (N-1)C_0 + NC_M \int_0^T r(t)\,dt + \dfrac{\epsilon N(N-1)C_M}{2}\,T^2}{NT} \qquad (3.125)$$

We may treat N as a continuous variable and set the derivative of (3.125) with respect to N to zero

$$\frac{\partial C(N,\overline{T})}{\partial N} \;=\; \frac{-C_R + C_0}{N^2 T} + \frac{\epsilon C_M}{2}\,T \;=\; 0$$

which yields

$$N = \frac{1}{T}\sqrt{\frac{2(C_R-C_o)}{\epsilon C_M}} \qquad (3.125)$$

Let

$$N' = \frac{1}{T_{N*}^*}\sqrt{\frac{2(C_R-C_o)}{\epsilon C_M}} \qquad (3.125)$$

Thus $N^* = [N']$ or $[N']+1$ depending on the minimum cost obtained from (3.108).

$$C(N,\overline{T}^*) = C_M[r(T_N^*) + \epsilon(N-1)T_N^*] \qquad (3.126)$$

Since T_1^* is the largest value among T_i^*'s, we can define a lower limit for N^*.

$$N^* \geq \left[\frac{1}{T_1^*}\sqrt{\frac{2(C_R-C_o)}{\epsilon C_M}}\right] \qquad (3.127)$$

For the Weibull distribution (3.127), this becomes

$$N^* \geq \left[\frac{\alpha\lambda C_M}{(\alpha+1)C_R}\right]^{1/(\alpha+1)}\sqrt{\frac{2(C_R-C_o)}{\epsilon C_M}} \qquad (3.128)$$

This estimate for N^* can save computation time required to evaluate the optimal solutions.

Let us evaluate equation (3.116) in case the underlying distribution is Weibull

$$\frac{\alpha\lambda}{\alpha+1} T_N^{\alpha+1} + \frac{\epsilon(N-1)}{2} T_N^2 = h(N) \qquad (3.129)$$

When $\alpha=1$ and 3 the analytical solutions can be obtained. For $\alpha=1$, we get

$$T_N^* = \left[\frac{2h(N)}{\lambda + \epsilon(N-1)} \right]^{1/2} \tag{3.130}$$

Similarly when $\alpha = 3$

$$T_N^* = \left[-\frac{\epsilon(N-1)}{3\lambda} + \sqrt{\frac{\epsilon^2(N-1)^2}{9\lambda^2} + \frac{4h(N)}{3\lambda}} \right]^{1/2} \tag{3.131}$$

Numerical Example 3.2

For a Weibull distribution

$$f(t) = \lambda t^\alpha \exp\left(-\frac{\lambda t^{\alpha+1}}{\alpha+1}\right) \qquad \alpha > 1; \quad \lambda > 0; \quad t > 0$$

with parameter $\lambda=1$, $\alpha=2$, and maintenance costs $C_R=15$, $C_o=5$, $C_M=1$, and the deterioration rate $=0.1$. We applied gradient technique to $C(N,\overline{T})$ and the optimum number of major repairs per cycle (including replacement) was found to be $N^*=7$ with average cost $C(7,T^*)=5.14$. The optimum time intervals are $T_i(7)=2.02$; $i=1,2,\ldots,7$. Figure 3.7 illustrates the behavior of $C(N,\overline{T}^*)$ as N is incremented. It has a decreasing and then increasing behavior and its minimum is unique. The lower bound using (3.128) yields $N^* \geq 6$.

Thus far we have assumed that $T_i=T_j=T_N$. In our numerical examples, this seems to be the only optimal solution. In the Weibull case with $\alpha=1$, this is the only solution. For $a \neq b$, we can see easily that

$$C(N,\overline{T})|T_i=a, \; t_j=b = C(N,\overline{T})|T_i=b, \; T_j=a \qquad i \neq j \tag{3.132}$$

which implies that equation (3.107) can have $N!+1$ solutions. If $C(N,\overline{T})$ has a unique minimum, then the only solution is $T_i=T_j$; $i \neq j$.
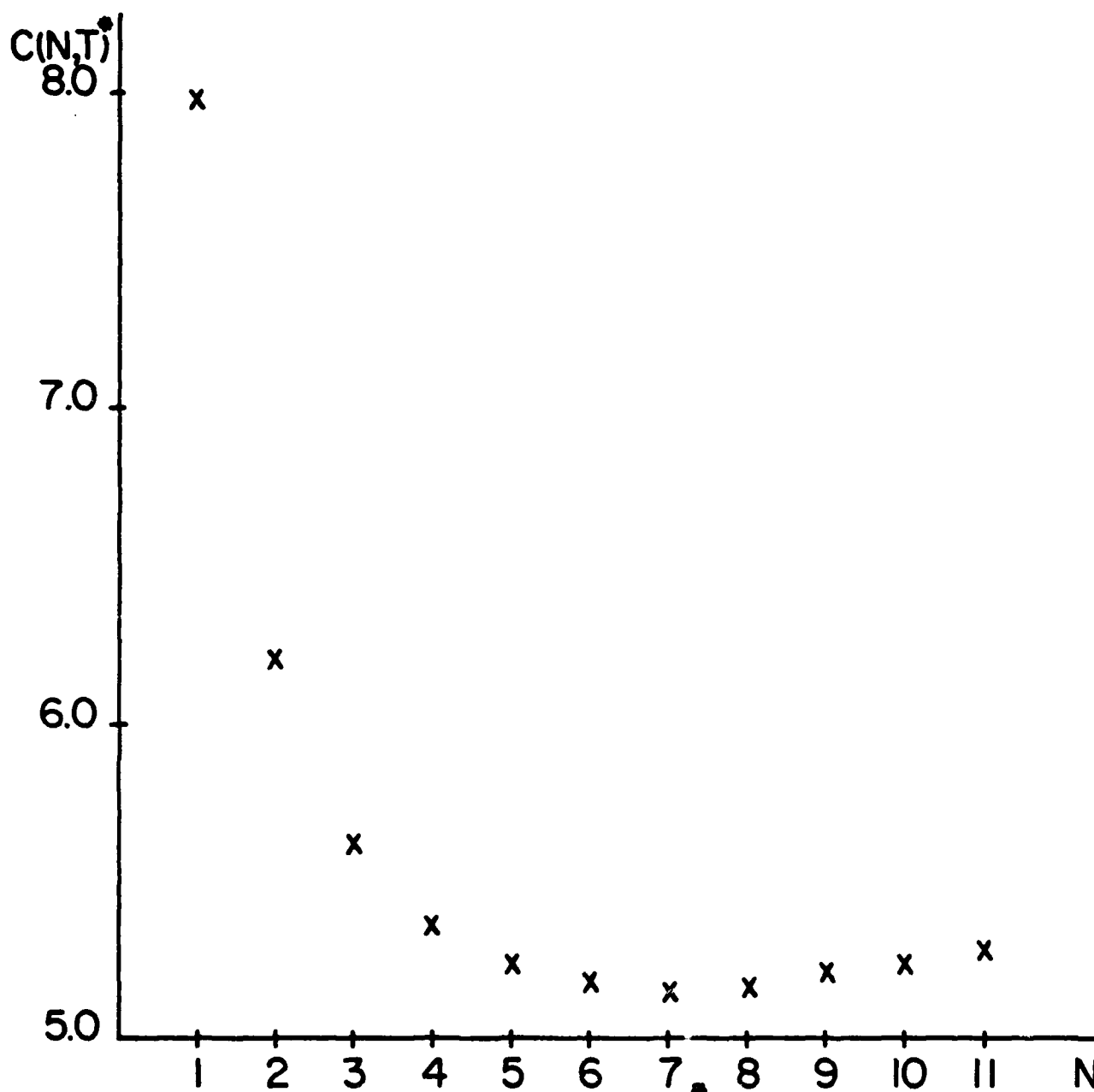
Figure 3.7 - Behavior of C(N,T) as N is
incremented for Policy II
(Model B)

3.12 AVAILABILITY

In our discussion on type II policy, we assumed that replacement, minimal repair, and major repair times possess zero duration. However, by interpreting C_R, C_M, and C_O as constant times for replacement, minimal repair and major repair costs, we can formulate the type II policy as steady state availability

$$A(N,\bar{T}) = \frac{\sum_{j=1}^{N} T_j}{\sum_{j=1}^{N} T_j + C_R + (N-1)C_O + C_M \sum_{j=1}^{N} \int_0^{T_j} r_j(t|\theta_{j-1})dt}$$

$$= [1 + C(N,\bar{T})]^{-1} \qquad (3.133)$$

It is seen easily that, from the optimization point of view, maximizing the availability is equivalent to minimizing the long-term average cost per unit time.

C H A P T E R   4

C O N C L U S I O N S

4.1 INTRODUCTION

We have studied the optimization of two major preventive mainte-
nance policies for repairable stochastically failing systems with lifetime
distribution function $F_i(t|\theta_{i-1})$ and failure rate $r_i(t|\theta_{i-1})$, where i
indicates the number of performed repairs, and the system's age at the
time of the $(i-1)^{st}$ repair is represented by t_{i-1}. Our objective for
each strategy is to optimize a set of successive maintenance intervals
$T_1, T_2, \ldots, T_N$ and the number of maintenance intervals N for each
renewal cycle so that the long-term average cost per unit time is min-
imized. Each preventive maintenance policy has been classified into two
different classes, namely, state- and age-dependent models. State-
dependent refers to the case when the failure rate at the time of the
i^{th} corrective maintenance is a function of previous number of repairs
or $r_i(t|\theta_{i-1}) = r_i(t)$. In the age-dependent model, the system's age at
the time of the ith repair will determine the $(i+1)^{st}$ failure rate. This
is usually shown by $r_i(t|\theta_{i-1}) = \theta_{i-1}r(t)$ where $\theta_{i-1} = 1 + \epsilon[\text{age}(i-1)]$.
ϵ is a constant positive parameter and is called age deterioration
factor. The underlying lifetime distribution is Weibull with a strictly
increasing failure rate.

Our goal has been efficiency, and in this context, many results
have been obtained, among which only a few are to be mentioned.

4.2 POLICY I

For state-dependent policy I, it was found that the computational algorithm developed by Nguyen and Murthy (1981), can be improved. This is done by finding a heuristic procedure. We can compute N'^* by which $C(N,\infty)$ is minimized, where N'^* provides a lower bound for N^*. This procedure is also applicable for an age-dependent in which the underlying lifetime distribution has the Weibull form.

The age-dependent policy I was optimized by simulation and approximation techniques. In both cases, using Weibull examples, it is observed that:

1. There is a unique N^* that minimizes the cost function $C(N,\overline{T}^*)$.

2. The optimum planned repair times are ordered as

$$T_1(N) > T_2(N) > \ldots > T_N(N)$$

and

$$T_1(1) > T_1(2) > \ldots > T_1(N^*) > T_1(N^*+1) > \ldots$$

Where in the state-dependent model, developed by Nguyen and Murthy (1981), $T_1(i)$; $i = 1,2,\ldots,N$ are ordered as

$$T_1(1) > T_1(2) > \ldots > T_1(N^*) < T_1(N^*+1) < \ldots$$

3. The results for simulation and approximation techniques seem to be similar but not identical.

4.3 POLICY II

For a general type II policy and a fixed N, it has been shown that the cost function $C(N,\overline{T})$ has a relative minimum and also that the optimal T_i satisfies

$$r_i(T_i^*|\theta_{i-1}) < r_j(T_j^*|\theta_{j-1}) \quad ; \quad i < j$$

These properties are useful when a optimum search technique is used.

The following results are valid only for Weibull lifetime distributions and A-type age-dependent policy II.

1. It is proved that the optimum major repair times are ordered as

$$T_1(N) > T(N) > \ldots > T_N(N)$$

2. Numerical examples have shown that the minimal cost, $C(N, \overline{T}^*)$, as a function of N, is decreasing for $N < N^*$ and increasing for $N > N^*$.

3. $\sum\limits_{i=1}^{N} T_i^*$, the total duration of an optimal renewal cycle, is an increasing function of N.

For B-type age-dependent policy II, *the following results seem to* be important:

1. One solution is $T_i^* = T_j^* = T_N^*$ for all i and j. This is proved sufficient when the age deterioration factor, , is not large and the underlying failure distribution is Weibull.

2. If the replacement cost is more than the major repair cost, then T_N^*, the optimal repair interval, is decreasing in N.

3. Our numerical example shows that the minimal cost $C(N, \overline{T}^*)$ is convex.

4. There exists a lower bound on N^*, which can be used as a good estimator.

5. Analytical solutions are obtainable for some Weibull cases.

6. If $C(N, \overline{T})$ has a unique minimum with respect to $\overline{T}$, then the only optimal solution is $T_i = T_j^*$; for all i,j.

4.4 PROBLEMS FOR FURTHER RESEARCH

A few possible extensions are as follows:

1. For age-dependent policy I, one may prove or determine conditions under which the optimal expected long-term average cost per unit time $C(N,\bar{T}^*)$, as a function of N, is decreasing for $N<N^*$ and increasing for $N>N^*$. To do so, equations (2.97), (2.103), and the behavior of the expected age of the system, $t_N^* = \sum_{i=1}^{N} y_i^*$, as N increases seem to be helpful.

2. One may consider extensions similar to (1), but using equations (3.51) and (3.52) for policy II (type A). In this case, the increasing property of $t_N^* = \sum_{i=1}^{N} T_i(N)$, as a function of N, must be proved. for type B of policy II, equation (3.116) is essential.

3. We have assumed that the replacement, major repair, minimal repair and breakdown costs are constant. One may take a more practical approach by defining these maintenance costs to be functions of elapsed operating time of the system. In this case, it might be easier to minimize the expected total cost rather than the expected long-term average cost per unit time.

4. Our models can be generalized by assuming non-zero and random down-times for replacement and repairs.

5. Another generalization is to have failure rates increasing in the number of major repairs and real age of the system simultaneously, where we have studied them separately.

6. Policy II can be modified by including a constant or minimum up-time constraint where, in policy I, average up-time may be restricted.

7.   It was assumed that the minimal repairs always restore the system to operation.   One may reformulate the policy II such that failures are corrected by minimal repairs with probability $p \geq 0$.

8.   It may be possible to find parameter sets for which the optimal policy permitting intermediate repairs has no such repairs.

APPENDIX

MULTIDIMENSIONAL MINIMIZATION BY CONJUGATE GRADIENT METHOD

Conjugate gradients are among the advanced search methods to calculate an unconstrained minimum of a real function with several variables. The underlying method is found in an article by R. Fletcher and C. M. Reeves (1964).

The method is used to find $\overline{X}^*$ (a column vector) which minimizes $C(\overline{X})$ by a sequence of moves from an initial point $\overline{X}_0$ to a new point $\overline{X}_1$, then to $\overline{X}_2$, and so on, where the gradient $\nabla C(\overline{X}) = G(\overline{X})$ is available analytically. Proceeding from an arbitrary initial search point $\overline{X}_0$, we locate a sequence of points that are successively closer to the minimum by the relation

$$\overline{X}_{i+1} = \overline{X}_i + \alpha_i \overline{S}_i$$

where $\overline{X}_i$ is the current point and α_i is a positive scalar that defines the distance between $\overline{X}_i$ and $\overline{X}_{i+1}$ along the search $\overline{S}_i$ vector (a column vector). Notice that the minimum along $\overline{S}_i$ will occur where the gradient of $C(\overline{X}_{i+1})$ will be normal to $\overline{S}_i$. Although the algorithm is supposed to find the optimum of a quadratic form, it is in fact applicable to a broader class of functions. The reason is that any function $C(\overline{X})$ in the neighborhood of the required minimum $\overline{X}^*$, can be approximated closely by the first three terms of its Taylor expansion and consequently by a quadratic form.

Given $\overline{X}_0$ and $G(\overline{X}_0)$, the conjugate gradient method can be described as follows:

STEP 0: Set $i = 0$; $\overline{S}_0 = -G(\overline{X}_0)$.

STEP 1: Form $C(\alpha_i) = C(\overline{X}_i + \alpha_i \overline{S}_i)$.

STEP 2: Find α_i^*, the value of α_i which minimizes $C(\alpha_i)$.

STEP 3: Obtain new point $\overline{X}_{i+1} = \overline{X}_i + \alpha_i \overline{S}_i$.

STEP 4: Determine new gradient $G(\overline{X}_{i+1})$.

STEP 5: Evalutae $C(\overline{X}_{i+1})$.

STEP 6: To test for optimality, STOP if either

      a. $\Delta C_i = C(\overline{X}_{i+1}) - C(\overline{X}_i) \geq 0$.

      b. $(G(\overline{X}_{i+1})^T G(\overline{X}_{i+1}) \leq \delta$.

      c. $i \geq n$.

STEP 7: Compute $\beta_i = \dfrac{(G(\overline{X}_{i+1}))^T G(\overline{X}_{i+1})}{(G(\overline{X}_i))^T G(\overline{X}_i)}$ -

STEP 8: Find the new search vector $\overline{S}_{i+1} = -G(\overline{X}_{i+1}) + \beta_i \overline{S}_i$.

STEP 9: Set $i = i+1$.

STEP 10: Go to STEP 1.

where $\delta > 0$ is a predetermined small number and n is the maximum allowable number of iterations. Notice that Step 2 usually requires a one-dimensional search.

We have used this algorithm to find the optimum scheduled repair times for age-dependent policy I (approximation and age-dependent policy II, where underlying lifetime distributions had Weibull forms. Despite the existence of nonnegativity contraint $\overline{T} \geq 0$, the conjugate gradient converges to the minimum point. Step 2 was carried out by employing the Golden Section search technique, where the initial interval of uncertainty for α_i was restricted to be positive.

Slightly less than 1.5 minutes of execution time was needed to find the optimum planned repair times for approximated age-dependent policy II with N=2. Using simulation, the same problem requires slightly greater computation time, but when N is large, simulation becomes more efficient than approximation. Much less execution time was needed for age-dependent policy II (Type A). For a typical example, less than one minute of computer time was spent to minimize $C(N,\bar{T})$ by finding $\bar{T}^*$ when N took on values from 1 to 20.

Here, we list FORTRAN subprograms written for approximated age-dependent policy I and type A of age-dependent policy II. It is assumed that the underlying lifetime distribution has Weibull form:

$$f_i(t|\theta_{i-1}) = \lambda_o \theta_{i-1} \exp(- \frac{\lambda_o \theta_{i-1} t^{\alpha+1}}{\alpha+1}); \; \lambda_o > 0, \; \alpha \geq 1, \; t > 0.$$

where θ_i is given by relations (2.69) and (3.75) for policies I and II respectively.

```
      SUBROUTINE  GRADII(N,T,EPS,CRNICO,CMLAM,ALPHA,C,G,TSUM)
      DIMENSION  T(N),G(N) ,COEFT(40)


C        GRADII ROUTINE EVALUATES THE GRADIENT OF THE EXPECTED LONG-TERM
C        AVERAGE COST FUNCTION C(N,T), WHEN THE AGE-DEPENDENT POLICY II
C        IS UNDER CONSIDERATION (SECTION 310). THE UNDERLYING FAILURE
C        DISTRIBUTION HAS THE WEIBULL FORM WITH ALPHA AND LAM AS SCALE
C        AND SHAPE PARAMETERS RESPECTIVELY. THE AGE FACTOR IS GIVEN BY
C        EQUATION (3.75).

C        VARIABLES THAT ARE GIVEN ARE:

C        N : NUMBER OF MAJOR REPAIRS INCLUDING REPLACEMENT IN A CYCLE.
C        T : A 1-DIM ARRAY OF SIZE N REPRESENTING THE SCHEDULED REPAIR TIMES.
C        EPS: AGE DETERIORATION FACTOR
C        CRNICO = CR+(N-1)CO : REPLACEMENT AND MAJOR REPAIR COSTS IN A CYCLE.
C        CMLAM = CM*LAM : (MINIMAL REPAIR)*(WEIBULL'S SCALE PARAMETER).

C        VARIABLES WILL BE RETURNED ARE :

C        C: COST FUNCTION C(N,T)
C        G : A 1-DIM ARRAY REPRESENTING THE GRADIENT OF C(N,T)


      ALPHA1 = ALPHA + 1.0
      TSUM = 0.0
      DO 10  I = 1,N
      TSUM = TSUM + T(I)
  10  CONTINUE
      D2 = 0.0
      DO  15 I = 2, N
      D2 = D2 + T(I)**ALPHA1
  15  CONTINUE
      COEFT(1) = 1.0
      M = N - 1
      DO 20 I = 1 ,M
      COEFT(I + 1) = COEFT(I) + EPS*T(I)
  20  CONTINUE
      D1 = 0.0
      DO 25 I = 1, N
      D1 = D1 + COEFT(I)*T(I)**ALPHA1
  25  CONTINUE
      R = CRNICO + CMLAM *D1/ALPHA1
      R1 = CMLAM*(T(1)**ALPHA + EPS*D2/ALPHA1)
      C = R/TSUM
      G(1) = (R1- C)/TSUM
      DO 30 I = 1 ,M
      R2 = COEFT(I+1)*T(I+1)**ALPHA-COEFT(I)*T(I)**ALPHA-EPS*T(I+1)**
     *ALPHA1/ALPHA1
      R2 = CMLAM * R2
      G(I + 1 ) = G(I) + R2/TSUM
  30  CONTINUE
      RETURN
      END
```

```
      SUBROUTINE GRADI (C)
      COMMON /BLT/T(20)/BLN/N/BLTH/THETA(20)/BLY/Y(20)/
     $ BLG/G(20)/BLL/LAMO/BLA/ALPHA/BLC/CR,CB/BLCO/CO/BLA1/ALPHA1/
     $ BLE/EPS
      DIMENSION V(20),FC(20),D(20,20),FCT(20),R(20)
      REAL IAMO,L(20)
      ALB=LAMO/ALPHA1
```

```
C        ROUTINE GRAD I EVALUATES THE GRADIENTOF THE EXPECTED LONG-TERM AVERAGE
C        COST C(N,T), WHEN APPROXIMATED AGE-DEPENDENT TYPE II POLICY (DISCUSSED
C        IN SECTION 2.9 ) IS APPLIED.
C        THE UNDERLYING FAILURE DISTRIBUTION HAS THE WEIBULL FORM GIVEN BY EQ.
C        2.116 AND THE AGE FACTOR IN THIS CASE IS DEFINED BY EQ. 2.117
C        OTHER VARIABLES ARE :
C
C        N : NUMBER OF MAJOR REPAIRS INCLUDING REPLACEMENT IN A CYCLE.
C        T : A 1-DIM ARRAY OF SIZE N REPRESENTING THE SCHEDULED REPAIR TIMES.
C        EPS: AGE DETERIORATION FACTOR
C        C: COST FUNCTION C(N,T)
C        G : A 1-DIM ARRAY REPRESENTING THE GRADIENT OF C(N,T)
C        ALPHA1 = ALPHA+1
C        AGE = SUM OF Y'S
C
C        VARIABLE              EQ. NO.
C
C           W                  (2.124)
C           Y                  (2.123)
C         THETA                (2.117)
C           D                  (2.112)
C           R                  (2.118)
C           L                  (2.114)
C           G                  (2.115)
```

```
      W=ALB**(1.0/ALPHA1)
      XR=W*T(1)
      Y(1)=AREA(1,XR)/W
      THETA(1)=1.0
      V(1)=AREA(2,XR)/W
      ALB=ALB*EPS
      DO 5 I=2,N
      W=W**ALPHA1+ALB*Y(I-1)
      W=W**(1.0/ALPHA1)
      XR=W*T(I)
      Y(I)=AREA(1,XR)/W
      THETA(I)=THETA(I-1)+EPS*Y(I-1)
      V(I)=AREA(2,XR)/(W*THETA(I))
    5 CONTINUE
      DO 10 I=1,N
      FC(I)=FBAR(T(I),I)
      D(I,I)=FC(I)
   10 CONTINUE
```

```
      N1=N-1
      DO 15 J=1,N1
      A1=0.0
      J1=J+1
      DO 15 I=J1,N
      A1=A1+D(I-1,J)
      D(I,J)=-EPS*A1*V(I)
15    CONTINUE
      DO 20 I=2,N
      FCT(I)=T(I)**ALPHA1*FC(I)
20    CONTINUE
      R(N)=CB*LAM0*THETA(N)*T(N)**ALPHA
      R(N)=R(N)*FC(N)
      DO 30 I=1,N1
      E=0.0
      ST=0.0
      DO 25 M=I,N1
      E=E+D(M,I)
      ST=ST+E*FCT(M+1)
25    CONTINUE
      ST=ST*ALR
      FT=LAM0*THETA(I)*T(I)**ALPHA
      FT=FT*FC(I)
      R(I)=CB*(FT+ST)
30    CONTINUE
      AGE=(THETA(N)-1.0)/EPS+Y(N)
      DO 35 I=1,N
      L(I)=0.0
      DO 35 J=I,N
      L(I)=L(I)+D(J,I)
35    CONTINUE
      B1=CB+(N-1)*C0+N*CB
      R2=0.0
      DO 40 I=1,N
      R2=R2+D(I,I)
40    CONTINUE
      R1=R1-CB*R2
      C=R1/AGE
      DO 45 I=1,N
      G(I)=R(I)-L(I)*C
      G(I)=G(I)/AGE
45    CONTINUE
      RETURN
      END
```

```
      FUNCTION  AREA(K,XR)
      DIMENSION  B(40,40)

C        AREA(K,X) PERFORMS NUMERICAL INTEGRATION IN ITERVAL (0,X) FOR
C        EXP(-X**(ALPHA+1))                        IF K=1
C        X**(ALPHA+1)*EXP(-X**(ALPHA+1))           IF  K=2


      DO 131 I=1,40
      DO 131 J=1,40
      B(I,J)=0.0
131   CONTINUE
      XL=0.0
      DEL=XR-XL
      B(1,1)=DEL*(FD(K,XL)+FD(K,XR))/2.0
      B(1,2)=B(1,1)+DEL*FD(K,(XL+XR)/2.0)
      B(1,2)=B(1,2)/2.0
      B(2,1)=4*B(1,2)-B(1,1)
      B(2,1)=B(2,1)/3.0
      J=3
5     D=DEL/(2**(J-1))
      X=XL-D
      NN=2**(J-2)
      SUM=0.0
      DO  10  I=1,NN
      X=X+2*D
      SUM=SUM+FD(K,X)
10    CONTINUE
      B(1,J)=B(1,J-1)/2.0+D*SUM
      L1=1
      DO  15  L=2,J
      KK=J-L+1
      L1=L1*4
      B(L,KK)=L1*B(L-1,KK+1)-B(L-1,KK)
      B(L,KK)=B(L,KK)/(L1-1)
15    CONTINUE
      B1=(B(J,1)-B(J-1,1))/B(J,1)
      B1=ABS(B1)
      IF  (B1.LT.1.0E-6)  GO  TO  20
      J=J+1
      GO  TC  5
20    AREA=E(J,1)
      RETURN
      END
```

```fortran
      FUNCTION  FBAR(X,K)
      COMMON/BLL/LAMO/BLA1/ALPHA1/BLTH/THETA(20)
      REAL LAMO


C        FBAR WILL RETURN THE VALUE OF
C        EXP(-X**(ALPHA+1)*LAMO*THETA/(ALPHA+1))

      X1=LAMO*THETA(K)/ALPHA1
      X2=X**ALPHA1
      X3=X1*X2
      IF(X3 .LT. 1.0-75) GO TO 1
      FBAR=EXP(-X3)
      RETURN
    1 FBAR=1.0
      RETURN
      END




      FUNCTION  FD(K,X)
      COMMON /BLA1/ ALPHA1


C        FD(K,X) WILL RETURN THE VALUE OF
C        EXP(-X**(ALPHA+1))                        IF K=1
C        X**(ALPHA+1)*EXP(-X**(ALPHA+1))           IF  K=2

      IF(X .LE. 1.0E-75 .AND. K .EQ. 2) GO TO 4
      IF(X .LE. 1.0E-75 .AND. K .EQ. 1) GO TO 5
      X1=X**ALPHA1
      IF(X1 .GT. 1.0E-75) GO TO 3
      X2=1.0
      GO TO 2
    3 IF(X1 .GT. 170.0) GO TO 4
      X2=EXP(-X1)
    2 IF(K.EQ.1)  GO  TO  1
      FD=X1*X2
      RETURN
    1 FD=X2
      RETURN
    4 FD=0.0
      RETURN
    5 FD=1.0
      RETURN
      END
```

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